MAT237 Multivariable Calculus with Proofs
Problem Set 6
Due Friday February 11, 2022 by 13:00 ET

Instructions

This problem set is based on Module F: Integrals (F7 to F9) and Module G: Integration methods (G1 to G3). Please read the Problem Set FAQ for details on submission policies, collaboration rules, and general instructions.

- **Problem Set 6 sessions are held on Tuesday February 8, 2022 in tutorial.** You will work with peers and get help from TAs. Before attending, seriously attempt these problems and prepare initial drafts.
- **Submissions are only accepted by Gradescope.** Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

Academic integrity statement

Full Name: **Jace Alloway** ______

Student number: **1006940802** ______

Full Name: ______

Student number: ______

I confirm that:

- I have read and followed the policies described in the Problem Set FAQ.
- I have read and understand the rules for collaboration on problem sets described in the Academic Integrity subsection of the syllabus. I have not violated these rules while writing this problem set.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while writing this assessment.

By signing this document, I agree that the statements above are true.

Signatures: 1) Jace Alloway ______

2) ______

Created and revised by C. Jonker, T. Janisse, and A. Zaman. Last updated February 11, 2022 at 15:45:05.

Problems

1. Let S be a Jordan measurable set in $\mathbb{R}^n$. Let f be a bounded real-valued function on $\overline{S}$.

   Show that if f is integrable on S, then f is integrable on $\overline{S}$ and $\int_S f\,dV = \int_{\overline{S}} f\,dV$. *Hint:* $\chi_{\overline{S}} = \chi_S + (\chi_{\overline{S}} - \chi_S)$

   (Revised 2022-02-07) Aside from Theorem 6.7.8, do not use any other theorems or lemmas from Section 6.7.

Notice that $\overline{S} = S^o \cup \partial S$. By **Lemma 2.2.21**, $S^o \cap \partial S = \varnothing$. Thus by **Lemma 6.6.14**,

$$\mathrm{vol}(\overline{S}) = \mathrm{vol}(S^o \cup \partial S) = \mathrm{vol}(S^o) + \mathrm{vol}(\partial S) - (S^o \cap \partial S) = \mathrm{vol}(S^o) + \mathrm{vol}(\partial S)$$

since $\mathrm{vol}(\varnothing) = 0$ trivially. By assumption, since S is Jordan measurable, then the boundary of S, ∂S has zero Jordan measure. It then follows by **Lemma 6.6.15** that $\mathrm{vol}(\partial S) = 0$, and thus

$$\mathrm{vol}(\overline{S}) = \mathrm{vol}(S^o).$$

Now $S^o \subseteq S \subseteq \overline{S}$ by **Lemma 2.2.10**, **Lemma 2.2.31**. It then follows that by **Lemma 6.6.14**,

$$\mathrm{vol}(S^o) \le \mathrm{vol}(S) \le \mathrm{vol}(\overline{S}).$$

However, since we proved that $\mathrm{vol}(\overline{S}) = \mathrm{vol}(S^o)$, then it must be that $\mathrm{vol}(S) = \mathrm{vol}(\overline{S})$. Now, since f is bounded, then there exists an $M > 0$ such that $|f| \le M$. Furthermore, since S is Jordan measurable, S is bounded and so there exists a rectangle R which bounds S (and hence $\overline{S}$). Since f is integrable on S, by definition the function $\chi_S f : \mathbb{R}^n \to \mathbb{R}$ is integrable on R. Then

$$-M\chi_S \le \chi_S f \le M\chi_S \implies -M\int_R \chi_S\,dV \le \int_R \chi_S f\,dV \le M\int_R \chi_S\,dV.$$

And thus

$$-M\,\mathrm{vol}(\overline{S}) = -M\,\mathrm{vol}(S) \le \int_R \chi_S f\,dV \le M\,\mathrm{vol}(S) = M\,\mathrm{vol}(\overline{S}),$$

which implies

$$\int_R \chi_S f\,dV = \int_R \chi_{\overline{S}} f\,dV.$$

By **Definition 6.7.1**, this shows that f is integrable on $\overline{S}$ and

$$\int_S f\,dV = \int_{\overline{S}} f\,dV.$$

2. Let m be the mass of a *non-rectangular* solid $S \subseteq \mathbb{R}^n$ with variable density $\delta : S \to [0, \infty)$. Assume $S \subseteq \mathbb{R}^n$ is Jordan measurable and δ is integrable on S. Use Riemann sums to heuristically justify the mass formula

$$m = \int_S \delta \, dV$$

as a limit of discrete approximations. Your argument will not be a formal proof, but be as precise as you can. (Revised 2022-02-01)

Since S is Jordan measurable, by definition S is bounded and its boundary ∂S has zero Jordan measure. Let R be the rectangle which bounds S. By **Theorem 6.6.9** since $S \subseteq R$, χ_S is integrable on R.
Let I be a finite set of multi indices and let $P = \{R_i\}_{i \in I}$ be a partition of R. Now choose and fix any sample point x_i^* for each R_i where $i \in I$, and thus the sum

$$m_P = \sum_{i \in I} \chi_S \delta(x_i^*) \cdot \mathrm{vol}(R_i) \approx m$$

approximates the mass of S. Let P' be a refinement of P, so $P \subseteq P'$. It follows that $m_{P'}$ will better approximate the mass of S, since the number of subrectangles increases. This argument follows from **Lemma 6.2.7**, however instead of choosing the infimum or supremum of δ on the subrectangle R_i', we choose and fix a sample point $x_i^{*\prime} \in R_i'$. Then

$$m_{P'} = \sum_{i \in I'} \chi_S \delta(x_i^{*\prime}) \cdot \mathrm{vol}(R_i') \approx m \qquad \text{and} \qquad 0 < |m - m_{P'}| < |m - m_P|.$$

To better approximate m, we can take finer and finer refinements of R, which will yield much less error with the exact value of the mass of S. Essentially we can let the number of partitions approach infinity, however it is more formal to say that for each subrectangle R_i'' of a refinement $P \subseteq P''$, $\|R_i''\| \to 0$. This is equivalent to

$$m = \lim_{\|R_i''\| \to 0} \sum_{i \in I''} \chi_S \delta(x_i^{*\prime\prime}) \cdot \mathrm{vol}(R_i'') = \int_R \chi_S \delta \, dV = \int_S \delta \, dV,$$

and so the mass formula is justified.

3. (Revised 2022-02-01) Define $\Omega = [-237, 237]^3$. Let $(\Omega, \Sigma, \mathbb{P})$ be a continuous probability space in $\mathbb{R}^3$ with probability density function ϕ. Define the set

$$S = \{(a, b, c) \in \Omega : b^2 - 4ac = 0\}.$$

(3a) Choose $(A, B, C) \in [-237, 237]^3$ randomly according to your continuous probability space $(\Omega, \Sigma, \mathbb{P})$ and define the quadratic form $q : \mathbb{R}^2 \to \mathbb{R}$ by

$$q(x, y) = Ax^2 + Bxy + Cy^2.$$

Note q has a critical point at $(0, 0)$. Assuming that S is an event in Σ and $\mathbb{P}(S) = 0$, explain why the second derivative test applied to q at $(0, 0)$ will be conclusive with probability one.

Since q is a polynomial, q is a C^∞ function on $\mathbb{R}$. The Hessian matrix of q is given by $Hq(x, y) = \begin{bmatrix} \partial_x^2 q & \partial_x \partial_y q \\ \partial_y \partial_x q & \partial_y^2 q \end{bmatrix}(x, y) = \begin{bmatrix} 2A & B \\ B & 2C \end{bmatrix}$. Now the determinant of the Hessian of q is

$$\det(Hq) = \det\left(\begin{bmatrix} 2A & B \\ B & 2C \end{bmatrix}\right) = 4AC - B^2.$$

Assuming that S is an event in Σ such that $\mathbb{P}(S) = 0$, it must be that for any $A, B, C \in \Omega$, $B^2 - 4AC \neq 0$ (since you will never measure this event occurring). This assumption implies that

$$\det\left(\begin{bmatrix} 2A & B \\ B & 2C \end{bmatrix}\right) = 4AC - B^2 \neq 0,$$

and thus the eigenvalues of the Hessian of q are never zero, since the determinant of an $n \times n$ matrix is the product of it's eigenvalues. By the second derivative test **(Theorem 5.5.9)**, $q(0, 0)$ will either have a local minimum or a local maximum, and thus the second derivative will always be conclusive (hence with probability one).

(3b) Prove that if S has zero Jordan measure, then S is an event in Σ and $\mathbb{P}(S) = 0$. *Hint:* Use two results.

Proof. Since S has zero Jordan measure, it is trivially Jordan measurable. Furthermore, since $S \subseteq \Omega = [-237, 237]^3$, then by definition of event space, $\Sigma = \{A \subseteq \Omega : \text{A is Jordan Measurable}\}$ **(Theorem 6.9.3)**, so S is an event in Σ.

Let $\phi : \Omega \to [0, \infty)$ be the probability density function of $\mathbb{P}$, and assume ϕ is bounded on Ω. By definition **(Theorem 6.9.7)**,

$$\mathbb{P}(S) = \int_S \phi \, dV.$$

Since ϕ is bounded on Ω and S has zero Jordan measure, then by **Theorem 6.7.8**,

$$\mathbb{P}(S) = \int_S \phi \, dV = 0.$$

$\square$

(3c) Finish the proof by showing S has zero Jordan measure. *Hint:* Proceed by definition. Cover the origin in $\mathbb{R}^3$ with a small rectangle depending on ε and *then* parametrize the remaining pieces.

Proof. It suffices to show that the parametric surface $C = \{(u, v, 2\sqrt{uv}) \in \mathbb{R}^3 : (u, v) \in [-237, 237]^2\}$ has zero Jordan measure. Fix $\varepsilon > 0$. Define the rectangle centered at $(0,0)$ by $[-f(\varepsilon), f(\varepsilon)]^3$. Note that C is bounded by the rectangle $U = [-237, 237]^2 \times [-474, 474]$. Choose $N =$ ___. The real part of C is located in the first and third quadrants, so I will cover the first quadrant with $4N$ rectangles, and hence I will use $8N$ rectangles to cover the whole region. It suffices to examine the first quadrant, since the identical process of proving C has zero Jordan measure can be applied to C in the third quadrant. Define the width of each subrectangle along then x, y and z axes as

$$\frac{237 - f(\varepsilon)}{N}.$$

Let $0 \le i, j \le N$ along each x, y axis respectively. Define each rectangle in the first quadrant by

$$R_{ij} = \left[f(\varepsilon) + \frac{237 - f(\varepsilon)}{N}(i-1), f(\varepsilon) + \frac{237 - f(\varepsilon)}{N}\, i\right] \times \left[f(\varepsilon) + \frac{237 - f(\varepsilon)}{N}(j-1), f(\varepsilon) + \frac{237 - f(\varepsilon)}{N}\, j\right]$$
$$\times \left[f(\varepsilon) + \frac{474 - f(\varepsilon)}{N}\sqrt{(i-1)(j-1)}, f(\varepsilon) + \frac{474 - f(\varepsilon)}{N}\sqrt{ij}\right].$$

For any $(u, v) \in [-237, 237]^2$, there exists an $i, j \in \{1, \ldots, N\}$ such that $(u, v) \in R_{ij}$, and thus $(u, v, 2\sqrt{uv}) \in R_{ij}$. Therefore C is contained in the union of every rectangle R_{ij}. Then

$$\operatorname{vol}(R_{ij}) = \left(\frac{237 - f(\varepsilon)}{N}\right)^2 \cdot \left(\frac{474 - f(\varepsilon)}{N}(\sqrt{(i-1)(j-1)} - \sqrt{ij})\right)$$
$$\implies \sum_{i,j=1}^{N} \operatorname{vol}(R_{ij}) = \left(\frac{237 - f(\varepsilon)}{N}\right)^2 \cdot \left(\frac{474 - f(\varepsilon)}{N}\right) \sum_{i,j=1}^{N} \left(\sqrt{(i-1)(j-1)} - \sqrt{ij}\right).$$

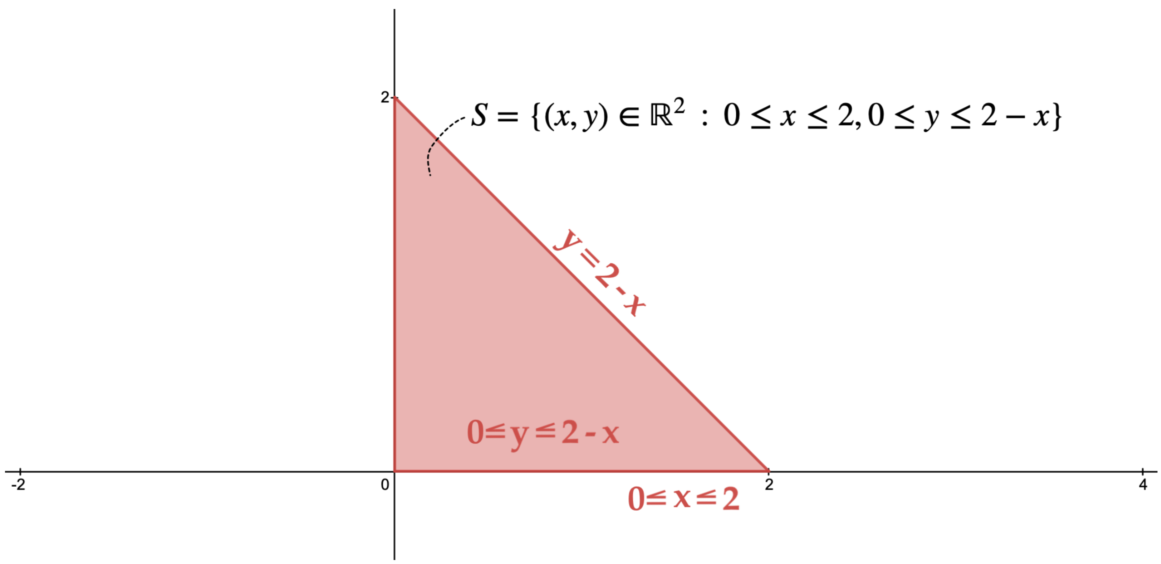
??.

4. Consider the integral

$$I = \int_0^2 \int_0^{2-x} (x + 2y)\, dy\, dx.$$

You will compute I by three different methods. Do not justify Fubini's theorem for any part below.

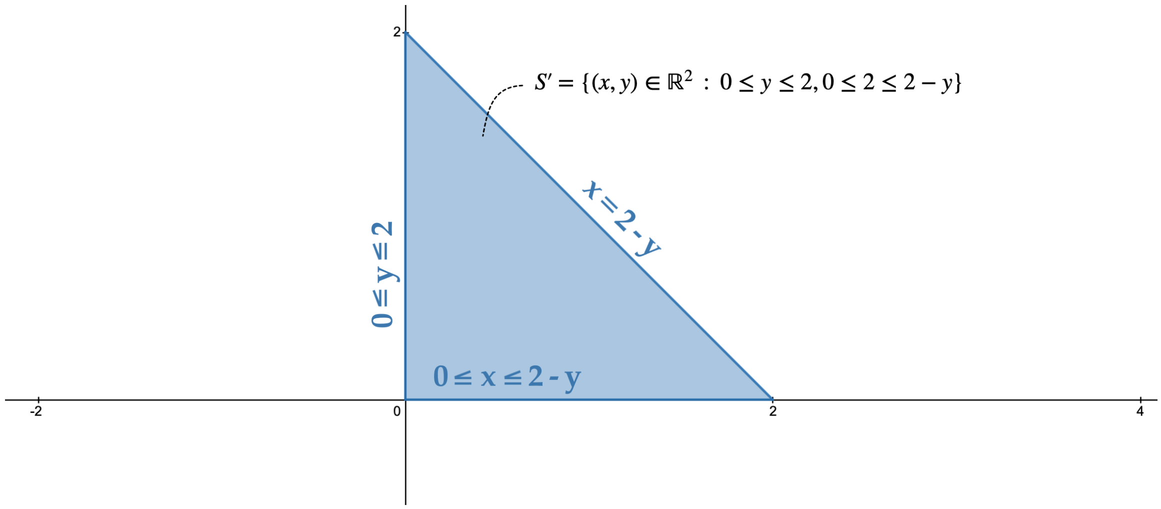
(4a) Sketch the region of integration, label it, and evaluate I directly. Do not use (4b) or (4c).



Evaluating,

$$\begin{aligned}
I &= \int_0^2 \int_0^{2-x} (x + 2y)\, dy\, dx \\
&= \int_0^2 \left[xy + y^2\right]\Big|_{y=0}^{y=2-x} dx \\
&= \int_0^2 x(2-x) + (2-x)^2\, dx \\
&= \int_0^2 (2x - x^2 + 4 + x^2 - 4x)\, dx \\
&= \int_0^2 (4 - 2x)\, dx \\
&= \left[4x - x^2\right]\Big|_{x=0}^{x=2} \\
&= [8-4] - [0-0] \\
&= 4.
\end{aligned}$$

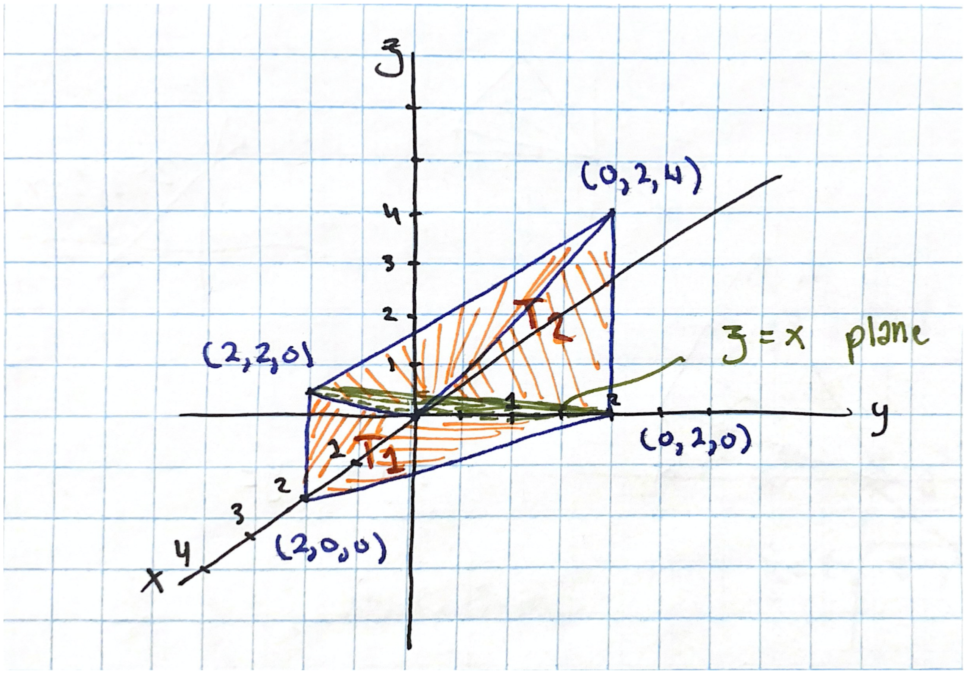
(4b) Swap the order of integration in I, sketch the region again with new labels, and evaluate the new double integral directly. Do not use (4a) or (4c).



Swapping the order of integration from $0 \leq x \leq 2, 0 \leq y \leq 2-x$ gives $0 \leq y \leq 2, 0 \leq x \leq 2-y$. I then becomes

$$
\begin{aligned}
I &= \int_0^2 \int_0^{2-y} (x+2y)\,dx\,dy \\
&= \int_0^2 \left[\frac{1}{2}x^2 + 2xy\right]\Big|_{x=0}^{x=2-y} dy \\
&= \int_0^2 \left[\frac{1}{2}(2-y)^2 + 2y(2-y)\right] dy \\
&= \int_0^2 \left[2 + \frac{1}{2}y^2 - 2y + 4y - 2y^2)\right] dy \\
&= \int_0^2 \left[-\frac{3}{2}y^2 + 2y + 2\right] dy \\
&= \left[-\frac{1}{2}y^3 + y^2 + 2y\right]\Big\|_{y=0}^{y=2} \\
&= [-4+4+4] - [-0+0+0] \\
&= 4.
\end{aligned}
$$

(4c) Interpret I as the volume of a solid. Include sketches in $\mathbb{R}^3$ and projections to support your argument. Compute the volume using geometry. Do not use (4a) or (4b).
Hint: The volume of a tetrahedron spanned by $u, v, w \in \mathbb{R}^3$ is given by $\frac{1}{6}\left|\det\left(\begin{bmatrix} u & v & w \end{bmatrix}\right)\right|$.



I can be interpreted as the area underneath the plane $z = x + 2y$ restricted to the domain

$$S = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 2,\ 0 \le y \le 2 - x\}$$

located in the xy plane. The corners of the solid can be represented using four vectors in $\mathbb{R}^3$:

$$\mathbf{u_1} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{u_3} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \mathbf{u_1} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}.$$

The volume of the solid can be divided by the plane $z = x$ into two separate tetrahedrons, T_1 with vertices $\mathbf{u_1}, \mathbf{u_2}$, and $\mathbf{u_3}$ and T_2 with vertices $\mathbf{u_2}, \mathbf{u_3}$, and $\mathbf{u_4}$. Therefore both tetrahedrons are spanned by their respective vectors.
Notice that the interiors of T_1 and T_2 are disjoint (the boundaries have zero volume), which allows us to calculate the volumes of T_1 and T_2 individually, then add them to get the total volume of the solid. Then

$$\mathrm{vol}(T_1) = \frac{1}{6}\det\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 2 & 4 & 0 \end{pmatrix} = \frac{1}{6} \cdot 8 \qquad \text{(by Wolfram Alpha)}$$

$$\mathrm{vol}(T_2) = \frac{1}{6}\det\begin{pmatrix} 2 & 2 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{pmatrix} = \frac{1}{6} \cdot 16 \qquad \text{(by Wolfram Alpha)}$$

$$\implies \mathrm{vol}(I) = \mathrm{vol}(T_1) + \mathrm{vol}(T_2) = \frac{1}{6}(8 + 16) = \frac{24}{6} = 4.$$

This is equivalent to the integrals computed in **(a)** and **(b)**.

5. Cavalieri's principle extends the idea of computing a volume by slices to higher dimensions. You will critique a proof that does not have any errors but it is missing details.

Theorem C. *Let S be a Jordan measurable set of $\mathbb{R}^{n+1}$ with $S \subseteq R \times [a,b]$ where R is a rectangle in $\mathbb{R}^n$. Assume the t-slice $S_t = \{x \in \mathbb{R}^n : (x,t) \in S\}$ is Jordan measurable for every $t \in [a,b]$. Then*

$$\operatorname{vol}(S) = \int_a^b \operatorname{vol}(S_t)\,dt.$$

(5a) Below are the first two lines of the proof for $n=1$ in which case $S \subseteq \mathbb{R}^2$.

> 1. *By definition,* $\operatorname{vol}(S) = \int_{R\times[a,b]} \chi_S \, dV$. *Write* $R = [c,d]$.
> 2. *By Fubini's theorem,* $\operatorname{vol}(S) = \int_a^b \int_c^d \chi_S(x,t)\,dx\,dt$

Line 2 applies Fubini's theorem without proper justification. Briefly justify it.

S is bounded by $R \times [a,b]$. It then follows that χ_S is also bounded on $R \times [a,b]$. Note that the set $S_t = \{x \in \mathbb{R}^n : (x,t) \in S\}$ is Jordan measurable for every $t \in [a,b]$, which implies that the indicator function $\chi_{S_t}(x)$ is integrable for any $x \in R = [c,d]$. That is, for a fixed $t \in [a,b]$,

$$\operatorname{vol}(S_t) = \int_R \chi_{S_t}\,dx = \int_c^d \chi_{S_t}\,dx.$$

Note that this holds for any t, and therefore Fubini's theorem applies. Thus

$$\operatorname{vol}(S) = \int_a^b \int_c^d \chi_S(x,t)\,dx\,dt$$

exists and is equivalent to

$$\operatorname{vol}(S) = \int\int_{R\times[a,b]} \chi_S\,dV.$$

(5b) The proof is completed with two more lines.

> 3. *For $x \in \mathbb{R}$ and $t \in \mathbb{R}$, notice* $\chi_S(x,t) = \chi_{S_t}(x)$.
> 4. *Thus,* $\operatorname{vol}(S) = \int_a^b \left(\int_c^d \chi_{S_t}(x)\,dx\right)dt$ — *by line 3*
>
> $= \int_a^b \left(\int_R \chi_{S_t}\,dV\right)dt$ — *by Fubini's theorem*
>
> $= \int_a^b \operatorname{vol}(S_t)\,dt.$ — *by definition of volume of a set*

Line 5 should cite its justifications line-by-line. For each of the three lines, add a short phrase on the righthand side explaining how it follows. You may cite a previous line, an assumption, a theorem, or a definition. Some details may still be missing but your phrase should indicate the key idea.

(5c) Use Cavalieri's principle to compute the volume of the 4-dimensional unit ball. You may assume the formula for the volume of 3-dimensional balls. You may assume 4-dimensional balls are Jordan measurable, but you may not assume 3-dimensional balls are Jordan measurable. (Revised 2022-02-07)

We assume the volume of a 3-ball (sphere) is given by $V_3 = \frac{4}{3}\pi r^3$. Assumed the 4-ball is a Jordan measurable set. Define the rectangle $R = [-1,1]^3$. The unit sphere is contained in R, and likewise, the unit hypersphere in 4 dimensions is contained in $R \times [-1,1]$. Define the unit 4-ball as the set

$$B_1^4 = \{x \in \mathbb{R}^4 : \|x\| \leq 1\}.$$

Then, by **Theorem C** since we assumed B_1^4 is Jordan measurable and the volume of the 3-ball is given by $V_3 = \frac{4}{3}\pi r^3$,

$$\operatorname{vol}(B_1^4) = \int_{R\times[-1,1]} \chi_{B_1^4}\, dV = \int_{-1}^{1} \frac{4}{3}\pi w^3\, dx.$$

Since the radius of the 4-ball is fixed, $1 = r = \sqrt{x^2+w^2}$ then we wish to integrate every slice over it, so $w = \sqrt{r^2-x^2} = \sqrt{1-x^2}$,

$$\operatorname{vol}(B_1^4) = \int_{-1}^{1} \frac{4}{3}\pi(1-x^2)^{3/2}\, dx.$$

By Wolfram Alpha,

$$\begin{aligned}
\operatorname{vol}(B_1^4) &= -\frac{4\pi(\sqrt{1-x^2}(2x^3-5x)-3\arcsin(x))}{24}\Big|_{-1}^{1} \\
&= \frac{\pi}{6}\left[\left(0+3\cdot\frac{\pi}{2}\right)-\left(0-3\cdot\frac{\pi}{2}\right)\right] \\
&= \frac{\pi^2}{2}.
\end{aligned}$$

And thus the volume of the unit 4-ball is $\frac{\pi^2}{2}$.