Q1) a) Let $y_1(t)=t$, $y_2(t)=t^2$, $y_3(t)=\frac{1}{t}$.

If y_1, y_2, y_3 fundamental, then $W[y_1, y_2, y_3] \neq 0$.

Computing:

$$W[y_1, y_2, y_3] = \det\begin{pmatrix} t & t^2 & \frac{1}{t} \\ 1 & 2t & -1/t^2 \\ 0 & 2 & 2/t^3 \end{pmatrix} = 2t^2 \cdot \frac{2}{t^3} + 0 + \frac{2}{t} - 0 - (2t)\left(\frac{-1}{t^2}\right) - 2\left(\frac{t^2}{t^3}\right)$$

$$= 4\frac{t^2}{t^3} + \frac{2}{t} + \frac{2t}{t^2} - \frac{2t^2}{t^3} = \boxed{\frac{6}{t} \neq 0}$$

and thus y_1, y_2, y_3 are a set of fundamental solutions.

b) The particular solution to the ODE

$$t^3y''' + t^2y'' - 2ty' + 2y = 2t^4$$

$$[\text{Normalize}] \longrightarrow y''' + \frac{1}{t}y'' - \frac{2}{t^2}y' + \frac{2}{t^3}y = 2t$$

is given by

$$y_p(t) = \sum_{m}^{k} y_m(t) \cdot \int_0^t \frac{g(s)\, W_m(s)}{W(s)}\, ds .$$

$$W_1(t) = \det\begin{pmatrix} 0 & t^2 & \frac{1}{t} \\ 0 & 2t & -\frac{1}{t^2} \\ 1 & 2 & \frac{2}{t^3} \end{pmatrix} \to \det\begin{pmatrix} t^2 & \frac{1}{t} \\ 2t & -\frac{1}{t^2} \end{pmatrix} = -\frac{t^2}{t^2} - \frac{2t}{t} = -3$$

$$W_2(t) = \det\begin{pmatrix} t & 0 & 1/t \\ 1 & 0 & -1/t^2 \\ 0 & 1 & 2/t^3 \end{pmatrix} \to -\det\begin{pmatrix} t & 1/t \\ 1 & -1/t^2 \end{pmatrix} = -\left(-\frac{t}{t^2} - \frac{1}{t}\right) = \frac{2}{t}$$

$$W_3(t) = \det\begin{pmatrix} t & t^2 & 0 \\ 1 & 2t & 0 \\ 0 & 2 & 1 \end{pmatrix} \to \det\begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix} = 2t^2 - t^2 = t^2$$

Then $y_p(t) = t \cdot \int_0^t \frac{2s(-3)}{\frac{6}{s}}\, ds + t^2 \int_0^t \frac{2s\left(\frac{2}{s}\right)}{\frac{6}{s}}\, ds + \frac{1}{t}\int_0^t \frac{2s(s^2)}{\frac{6}{s}}\, ds$

$$= -t\int_0^t s^2\, ds + \frac{2}{3}t^2\int_0^t s\, ds + \frac{1}{3}\frac{1}{t}\int_0^t s^4\, ds$$

$$= -\frac{1}{3}t^4 + \frac{1}{3}t^4 + \frac{1}{15}t^4$$

Thus $\boxed{y_p(t) = \frac{1}{15}t^4}$

c) Our general solution to the ODE is then

$$y(t) = c_1 t + c_2 t^2 + \frac{c_3}{t} + \frac{1}{15} t^4 .$$

$$\Rightarrow \quad y'(t) = c_1 + 2c_2 t - \frac{c_3}{t^2} + \frac{4}{15} t^3$$

$$\Rightarrow \quad y''(t) = 2c_2 + \frac{2c_3}{t^3} + \frac{12}{15} t^2$$

Then

$$\left.\begin{aligned} 1 &= c_1 + c_2 + c_3 + \tfrac{1}{15} \\ 1 &= c_1 + 2c_2 - c_3 + \tfrac{4}{15} \\ 1 &= 2c_2 + 2c_3 + \tfrac{12}{15} \end{aligned}\right\} \Rightarrow \begin{aligned} \tfrac{14}{15} &= c_1 + c_2 + c_3 \\ \tfrac{11}{15} &= c_1 + 2c_2 - c_3 \\ \tfrac{3}{15} &= 2c_2 + 2c_3 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 & \frac{14}{15} \\ 1 & 2 & -1 & 11/15 \\ 0 & 2 & 2 & 3/15 \end{bmatrix} \xrightarrow[r_1 - r_2]{r_3 \cdot \frac{1}{2}} \begin{bmatrix} 0 & -1 & 2 & 3/15 \\ 1 & 2 & -1 & 11/15 \\ 0 & 1 & 1 & 3/30 \end{bmatrix} \xrightarrow[r_2 - 2r_3]{r_1 + r_3} \begin{bmatrix} 0 & 0 & 3 & 9/30 \\ 1 & 0 & -3 & 8/15 \\ 0 & 1 & 1 & 3/30 \end{bmatrix}$$

$$\xrightarrow[r_2 + r_1]{r_1 \div 3} \begin{bmatrix} 0 & 0 & 1 & 1/10 \\ 1 & 0 & 0 & 5/6 \\ 0 & 1 & 1 & 3/30 \end{bmatrix} \xrightarrow{r_3 - r_1} \begin{bmatrix} 1 & 0 & 0 & 5/6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/10 \end{bmatrix} \qquad \begin{aligned} c_1 &= \tfrac{5}{6} \\ c_2 &= 0 \\ c_3 &= \tfrac{1}{10} \end{aligned}$$

Thus our solution to the IVP is

$$\boxed{y(t) = \frac{5}{6} t + \frac{1}{10 t} + \frac{1}{15} t^4}$$

MAT244 PS4

Q2) a) Our system is $x' = x - y$, $y' = x + y \Rightarrow y = x - x'$

$$\frac{d}{dx}\left[x' = x - y\right] \Rightarrow x'' = x' - y'$$
$$= x' - (x+y)$$
$$= x' - (x + (x - x'))$$
$$= 2x' - 2x$$

$\Rightarrow$ $\boxed{x'' - 2x' + 2x = 0}$, as required.

b) Our characteristic equation to the ODE is

$$x'' - 2x' + 2x = 0 \longrightarrow r^2 - 2r + 2 = 0$$

$$r = \frac{2}{2} \pm \frac{\sqrt{4 - 4(2)}}{2} = 1 \pm i$$

Our general solution is then

$$\boxed{x(t) = e^t\left(c_1 \cos(t) + c_2 \sin(t)\right)},$$

c) If $y = x - x'$, then

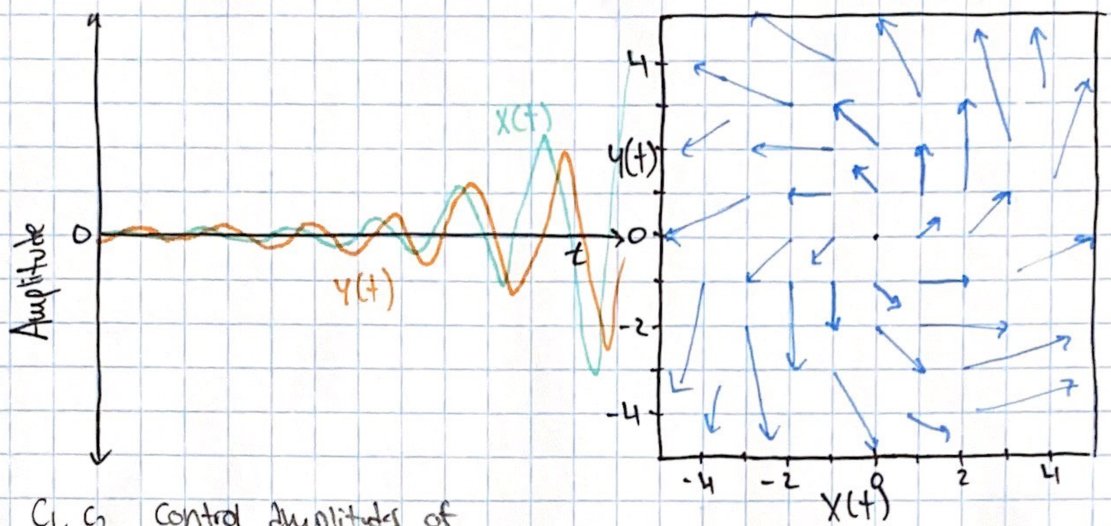
$$y(t) = e^t\left(c_1 \cos(t) + c_2 \sin(t)\right) - \left[e^t\left(c_1 \cos(t) + c_2 \sin(t)\right) + e^t\left(c_2 \cos(t) - c_1 \sin(t)\right)\right]$$

$$= e^t\left(c_2 \cos(t) - c_1 \sin(t)\right)$$

So

$$\boxed{\begin{aligned} x(t) &= e^t\left(c_1 \cos(t) + c_2 \sin(t)\right) \\ y(t) &= e^t\left(c_2 \cos(t) - c_1 \sin(t)\right) \end{aligned}}$$

d) Hand plots (I will include Python plots for more accuracy)



C_1, C_2 control amplitudes of each solution.

Solutions of the matrix $\vec{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \vec{x}$

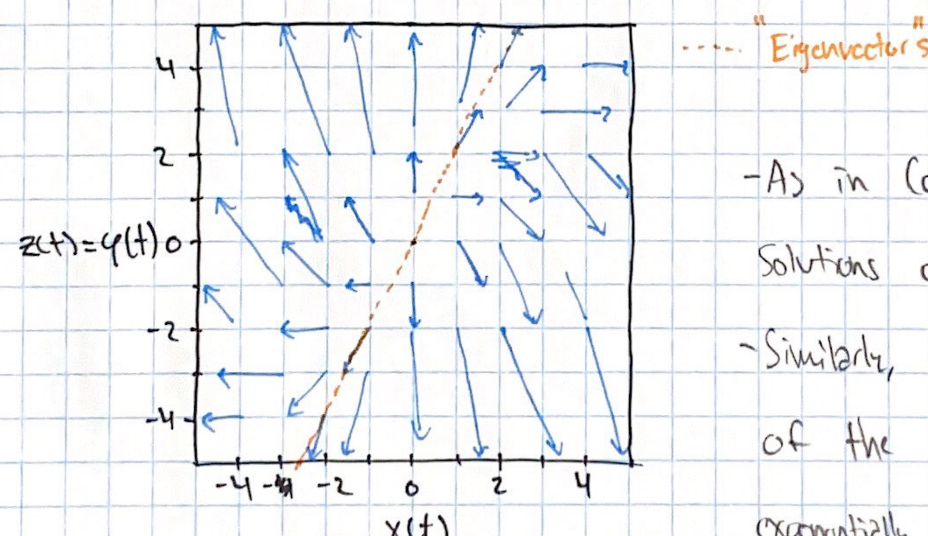
- As $t \to \infty$, the amplitude / vector magnitudes increase exponentially.

e) If we have the system

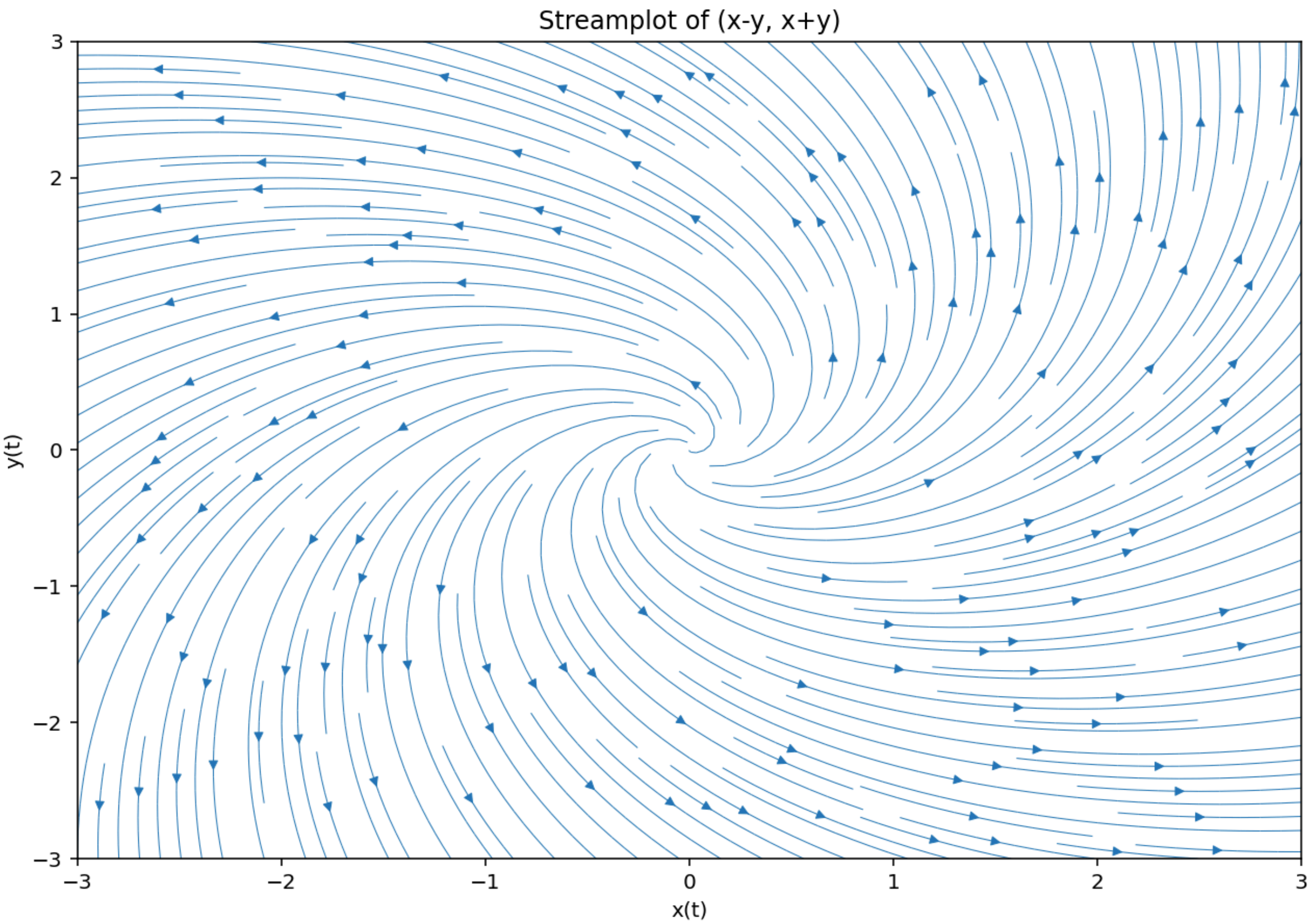
$$x' = z$$
$$z' = -2x + 2z \, ,$$

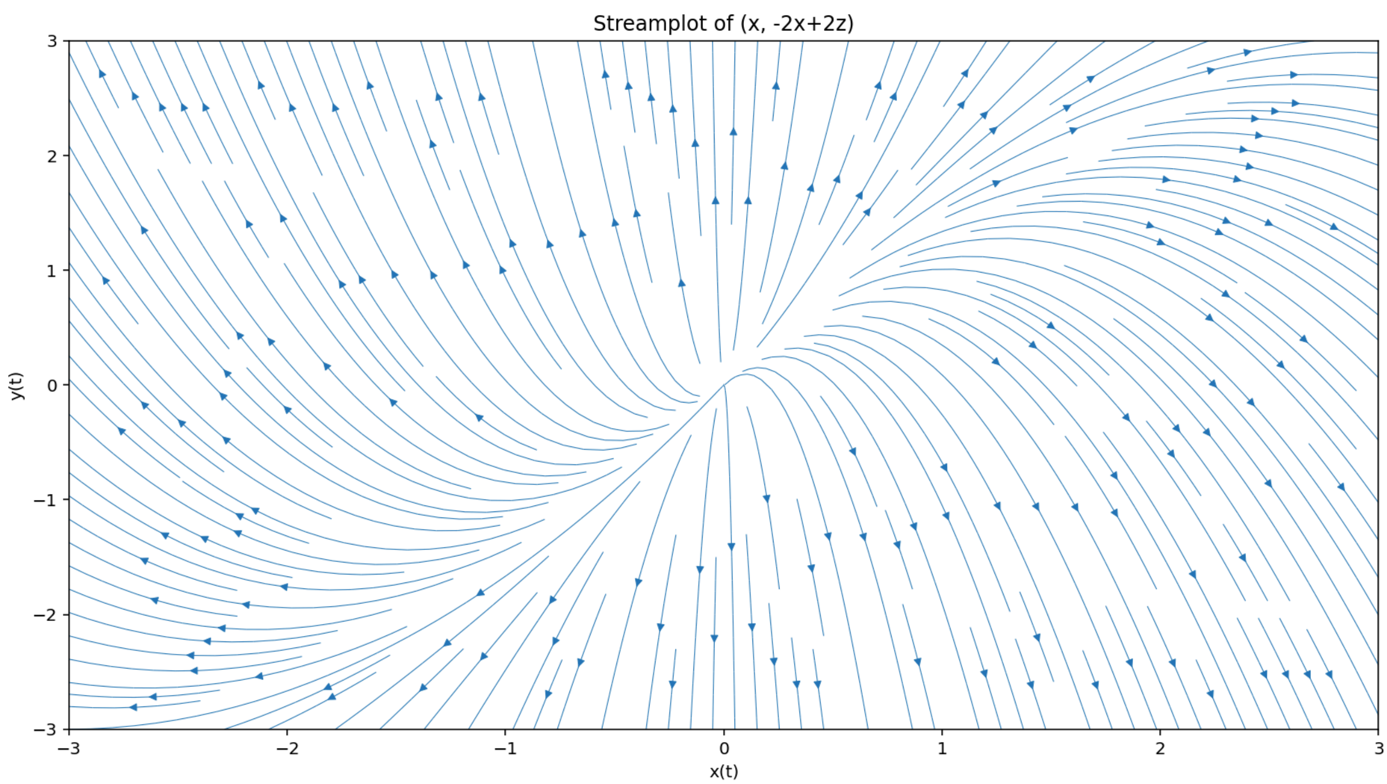
our matrix is $\begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}$.

Solutions plotted in the phase plane for $\vec{x}' = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \vec{x}$:



- As in (d), the solutions diverge.
- Similarly, the magnitude of the vectors increases exponentially


Streamplot of (x-y, x+y)
y(t)
x(t)
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3


Streamplot of (x, -2x+2z)
y(t)
x(t)
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3

MAT244 PS4

Q3) a) $A = \begin{pmatrix} -3 & -5 & 0 & 5 \\ 4 & 6 & 0 & -5 \\ 2 & 2 & 2 & -2 \\ 1 & 1 & 0 & 1 \end{pmatrix}$

Find: $char(A) = \det(A - \lambda I) = \det \begin{pmatrix} -3-\lambda & -5 & 0 & 5 \\ 4 & 6-\lambda & 0 & -5 \\ 2 & 2 & 2-\lambda & -2 \\ 1 & 1 & 0 & 1-\lambda \end{pmatrix}$

$= (-3-\lambda)\cdot\det\begin{pmatrix} 6-\lambda & 0 & -5 \\ 2 & 2-\lambda & -2 \\ 1 & 0 & 1-\lambda \end{pmatrix} - (-5)\cdot\det\begin{pmatrix} 4 & 0 & -5 \\ 2 & 2-\lambda & -2 \\ 1 & 0 & 1-\lambda \end{pmatrix} - 5\cdot\det\begin{pmatrix} 4 & 6-\lambda & 0 \\ 2 & 2 & 2-\lambda \\ 1 & 1 & 0 \end{pmatrix}$

$= (-3-\lambda)\cdot\left[(6-\lambda)(2-\lambda)(1-\lambda) + 0 + 0 - (-5)(2-\lambda) - 0 - 0\right]$

$+ 5\left[(4)(2-\lambda)(1-\lambda) + 0 + 0 - (1)(-5)(2-\lambda) - 0 - 0\right]$

$- 5\left[0 + (6-\lambda)(2-\lambda)(1) + 0 - 0 - (1)(2-\lambda)(4) - 0\right]$

$= (-3-\lambda)(6-\lambda)(2-\lambda)(1-\lambda) + 5(-3-\lambda)(2-\lambda) + 20(2-\lambda)(1-\lambda) + 25(2-\lambda)$

$- 5(6-\lambda)(2-\lambda) + 20(2-\lambda)$

$= (-18 - 3\lambda + \lambda^2)(2 - 3\lambda + \lambda^2) + 5(-6 + \lambda + \lambda^2) + 20(2 - 3\lambda + \lambda^2) + 25(2-\lambda)$

$- 5(12 - 8\lambda + \lambda^2) + 20(2-\lambda)$

$= -36 - 6\lambda + 2\lambda^2 + 54\lambda + 9\lambda^2 - 3\lambda^3 - 18\lambda^2 - 3\lambda^3 + \lambda^4 - 30 + 5\lambda + 5\lambda^2$

$+ 40 - 60\lambda + 20\lambda^2 + 50 - 25\lambda - 60 + 40\lambda - 5\lambda^2 + 40 - 20\lambda$

$= \lambda^4 - 6\lambda^3 + 13\lambda^2 - 12\lambda + 4$ We find $\lambda = 1, \lambda = 2 \Rightarrow char(A) = 0.$

- This is easy to factor, since if the only eigenvalues (hence with non-zero eigenvectors) are $\lambda = 1$, $\lambda = 2$, the the only combination of values $(\lambda-1)^n(\lambda-2)^m$ to create '4' are $n = m = 2$,

Thus $\boxed{char(A) = (\lambda-1)^2(\lambda-2)^2}$

$\begin{cases} \lambda = 1 \text{ with algebraic multiplicity } 2 \\ \lambda = 2 \text{ with algebraic multiplicity } 2. \end{cases}$

b) If λ is an eigenvalue to eigenvector $\vec{v}$, then by definition

$$A\vec{v} = \lambda\vec{v}, \quad \text{so}$$

$$\Rightarrow \quad A\vec{v} - \lambda\vec{v} = 0.$$

Factoring $\vec{v}$, we have $A\vec{v} - \lambda\vec{v} = \boxed{(A - \lambda I)\vec{v} = 0,}$

as required.

c) To find the geometric multiplicities, we need to find the corresponding eigenspaces:

$$\text{Ker}(A - 2I) = \text{Ker}\begin{pmatrix} -5 & -5 & 0 & 5 \\ 4 & 4 & 0 & -5 \\ 2 & 2 & 0 & -2 \\ 1 & 1 & 0 & -1 \end{pmatrix} \rightarrow \text{row reduce}$$

$$= \text{Ker}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{Let } \begin{matrix} z = t \\ w = s \end{matrix}, \quad x_1 = t\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad x_2 = s\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Therefore the basis of eigenvectors with eigenvalue $\lambda = 2$ is

$$\boxed{\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}}, \quad \text{which has dimension 2.}$$

Thus the algebraic & geometric multiplicities coincide for $\lambda = 2$

Note: the geometric multiplicity for the eigenvalue $\lambda = 1$ is 1.

d) To find the dimension of the solution space of A, we find the dimension of the image of A (as a map), or rank(A).

$$\begin{pmatrix} -3 & -5 & 0 & 5 \\ 4 & 6 & 0 & -5 \\ 2 & 2 & 2 & -2 \\ 1 & 1 & 0 & 1 \end{pmatrix} \xrightarrow[r_3 - 2r_4]{r_1 + 3r_4,\ r_2 - 4r_4} \begin{pmatrix} 0 & -2 & 0 & 8 \\ 0 & 2 & 0 & -9 \\ 0 & 0 & 2 & -4 \\ 1 & 1 & 0 & 1 \end{pmatrix} \xrightarrow[r_2 \times \frac{1}{2}]{r_1 \times -\frac{1}{2},\ r_3 \times \frac{1}{2}} \begin{pmatrix} 0 & 1 & 0 & -4 \\ 0 & 1 & 0 & -\frac{9}{2} \\ 0 & 0 & 1 & -2 \\ 1 & 1 & 0 & 1 \end{pmatrix} \xrightarrow[r_4 - r_1]{r_2 - r_1}$$

$$\rightarrow \begin{pmatrix} 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & 5 \end{pmatrix}, \quad \text{and thus the } \boxed{\text{rank is 3.}}$$

Thus the dimension of the solution space is 3.

Note: this implies that $\dim(\text{Ker}(A)) = 1$ By rank-nullity THM

e) The dimension of the solution space of A is equal to the geometric multiplicity because the sum of the geometric multiplicities represents the dimension of the eigenspace.

Since $\dim(\mathrm{Im}(A)) \neq \dim(V) = 4$, then A is not diagonalizable, but in the solution space, all vectors live in $\mathbb{R}^3$ because they can both be created by linear combinations of vectors in the eigenbasis or the $\mathcal{E}$-basis (which I deem the initial basis of A). Either way, the set of all vectors in the solution space has dimension 3, and that is why the dimension of the solution space of A coincide with the geometric multiplicity of A.

MAT244 PS4

Q4) (A) $x' = x^2 + y^2$, $y' = -2xy$ corresponds to (I).

- $x' = x^2 + y^2 \geq 0$ for every value of x and y, which corresponds to (I) since every streamline arrow has a component pointing in the positive x direction. This is different than (II) and (III) because the streamline arrows have negative x components.

(B) $x' = \sin(y)$, $y' = \cos(x)$ corresponds to (III).

- Since $-1 \leq \sin(t) \leq 1$, $-1 \leq \cos(t) \leq 1$ for all $t \in \mathbb{R}$, we expect the solution to oscillate in some way. In (III), we can see "spirals" of oscillating solutions, which is much different than in (I) and (II).

(C) $x' = x + y$, $y' = -x - y$.

- This system corresponds to the linear map $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Every vector outputted by T $(T(\vec{v}))$ is scaled by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ along each component, yielding a set of streamlines which look like $y = -x$. The set of solutions to this system are lines, and thus this system must correspond to (II) (also by the process of elimination).

→ Thus $\boxed{(A) \to (I), \quad (B) \to (III), \quad \text{and} \quad (C) \to (II)}$

MAT244 PS4

Q5) a) The determinant function is multilinear, and thus by multiplying any column by λ yields $\lambda \cdot \det(A)$.

Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. Then

$$\det(A) = aei + bfg + cdh - gec - hfa - idb.$$

$$\Rightarrow \det\begin{pmatrix} \lambda a & b & c \\ \lambda d & e & f \\ \lambda g & h & i \end{pmatrix} = \lambda aei + bf\lambda g + c\lambda dh - \lambda gec - hf\lambda a - i\lambda db$$

$$= \underline{\lambda \cdot \det(A)}$$

$$\det\begin{pmatrix} a & \lambda b & c \\ d & \lambda e & f \\ g & \lambda h & i \end{pmatrix} = a\lambda ei + \lambda bfg + cd\lambda h - g\lambda ec - \lambda hfa - id\lambda b$$

$$= \underline{\lambda \cdot \det(A)}$$

$$\det\begin{pmatrix} a & b & \lambda c \\ d & e & \lambda f \\ g & h & \lambda i \end{pmatrix} = ae\lambda i + b\lambda fg + \lambda cdh - ge\lambda c - h\lambda fa - \lambda idb$$

$$= \underline{\lambda \cdot \det(A)},$$

as required.

b) $B = \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{pmatrix}$

$$\cdot \det(B) = yz^2 + zx^2 + xy^2 - x^2y - y^2z - z^2x$$

$$= yz^2 - yxz - xz^2 + x^2z - yz^2 + y^2x + xyz - x^2y$$

$$= (yz - yx - xz + x^2)(z-y)$$

$$\boxed{= (y-x)(z-x)(z-y),}$$

as desired.

c) $y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$, $y_3 = e^{r_3 t}$.

Then

$$W[y_1, y_2, y_3] = \det \begin{pmatrix} e^{r_1 t} & e^{r_2 t} & e^{r_3 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} & r_3 e^{r_3 t} \\ r_1^2 e^{r_1 t} & r_2^2 e^{r_2 t} & r_3^2 e^{r_3 t} \end{pmatrix}$$

$$= e^{(r_1+r_2+r_3)t}(r_2 r_3^2) + e^{(r_2+r_3+r_1)t}(r_3 r_1^2) + e^{(r_2+r_1+r_2)t}(r_1 r_2^2) - e^{(r_1+r_2+r_3)t}(r_1^2 r_2) - e^{(r_2+r_3+r_1)t}(r_2^2 r_3) - e^{(r_1+r_2+r_3)t}(r_3^2 r_1)$$

$$= e^{(r_1+r_2+r_3)t}\left[r_2 r_3^2 + r_3 r_1^2 + r_1 r_2^2 - r_1^2 r_2 - r_2^2 r_3 - r_3^2 r_1\right]$$

$$\boxed{= e^{(r_1+r_2+r_3)t}\left[(r_2-r_1)(r_3-r_1)(r_3-r_2)\right]} \quad \text{From (b).}$$

d) If r_1, r_2, r_3 are all different (ie $r_1 \neq r_2 \neq r_3$) then the wronskian (from (c)) will never be zero (e^x is never zero) and is zero if and only if $r_1 = r_2$, $r_1 = r_3$, or $r_3 = r_2$.

Thus since $W[y_1, y_2, y_3] \neq 0$, y_1, y_2, and y_3 are a set of fundamental solutions to the ODE.