MAT244 PS5

Q1) a) $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$.

$$\text{Char}(A) = (-\lambda)(3-\lambda)^2 + (2)^2(4) + (4)(2)^2 - (-\lambda)(4)^2$$
$$-(2)^2(3-\lambda) - (3-\lambda)(2)^2$$
$$= -\lambda^3 - 9\lambda + 6\lambda^2 + 16 + 16 + 16\lambda - 12 + 4\lambda - 12$$
$$+4\lambda$$
$$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8.$$

We find that $\lambda = -1,\ \lambda = 8 \Rightarrow \text{char}(A) = 0$.

$$\begin{array}{r|rrrr} 8 & -1 & 6 & 15 & 8 \\ & & -8 & -16 & -8 \\ \hline & -1 & -2 & -1 & 0 \end{array} \longrightarrow (\lambda-8)(-\lambda^2-2\lambda-1) = -(\lambda-8)(\lambda+1)^2.$$

Thus $\boxed{\lambda = 8,\ \lambda = -1 \text{ with an algebraic multiplicity of } 2.}$

$\lambda = -1$
$A + I$ $\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $y = s \quad z = t$, $2x = -s - 2t$

$\boxed{\left\{ \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}. \quad \lambda = -1 \text{ Geometric Multiplicity } 2}$

$\lambda = 8$
$A - 8I$ $\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \xrightarrow[r_2+r_1]{r_1-r_3} \begin{pmatrix} -1 & 6 & 1 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \xrightarrow{r_3+r_2} \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 4 & 2 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$\boxed{\left\{ \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\} \quad \lambda = 8 \text{ Geometric Multiplicity } 1}$

b) $B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}$

$$\text{char}(B) = (1-\lambda)^2(4-\lambda) + (-1)(1)(-3) + (1)(2)^2$$
$$- (-3)(1)(1-\lambda) - (-1)(2)(1-\lambda) - (1)(2)(4-\lambda)$$
$$= 4 + 4\lambda^2 - 8\lambda - \lambda - \lambda^3 + 2\lambda^2 + 4 - 3\lambda$$
$$= -\lambda^3 + 6\lambda^2 - 12\lambda + 8.$$

We find $\lambda = 2 \Rightarrow \text{char}(B) = 0$. $\boxed{\lambda = 2 \text{ with algebraic multiplicity } 3.}$

$$\begin{array}{r|rrrr} 2 & -1 & 6 & -12 & 8 \\ & & -2 & 8 & -8 \\ \hline & -1 & 4 & -4 & 0 \end{array}$$

$(\lambda-2)(-\lambda^2+4\lambda-4) = -(\lambda-2)^3$.

$$B-2I \quad \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \xrightarrow{r_3+2r_2} \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow[r_1+r_2]{r_1+r_3,\ r_2-2r_3} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$x=0$
$z=t$
$y=-t$

$\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$ $\lambda = 2$

$\lambda=2$ Geometric multiplicity 1

$\therefore$ Non diagonalizable.

c) $C = \begin{pmatrix} 5 & -3 & 2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix}$

$$\begin{aligned}
\text{char}(C) &= (5-\lambda)(-5-\lambda)(3-\lambda) + (-3)(-4)^2 + (2)(8)(3) - (-4)(-5-\lambda)(2) \\
&\quad - (3)(-4)(5-\lambda) - (3-\lambda)(8)(-3) \\
&= -75 + 3\lambda^2 + 25\lambda - \lambda^3 - 40 - 8\lambda + 60 - 12\lambda + 72 - 24\lambda \\
&= -\lambda^3 + 3\lambda^2 - 19\lambda + 17.
\end{aligned}$$

We find $\lambda = 1 \Rightarrow \text{char}(C) = 0.$

$$\begin{array}{r|rrrr} 1 & -1 & 3 & -19 & 17 \\ & & -1 & 2 & -17 \\ \hline & -1 & 2 & -17 & 0 \end{array}$$

$(\lambda - 1)(-\lambda^2 + 2\lambda - 17) = 0.$

$$\lambda = \frac{2}{2} \pm \frac{\sqrt{4 - 4(1)(17)}}{2} = 1 \pm 4i.$$

So $\boxed{\lambda = 1,\ \lambda = 1 \pm 4i \text{ all with algebraic multiplicity } 1}$

$$A - I \begin{pmatrix} 4 & -3 & 2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \boxed{\left\{ \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \right\} \ \lambda = 1, \text{ geometric multiplicity } 1.}$$

$A - (1+4i)I$

$$\begin{pmatrix} 4-4i & -3 & 2 \\ 8 & -6-4i & -4 \\ -4 & 3 & 2-4i \end{pmatrix} \begin{matrix} \cdot(1+i) \\ \\ \cdot 2 \end{matrix} \longrightarrow \begin{pmatrix} 8 & -3-3i & 2+2i \\ 8 & -6-4i & -4 \\ -8 & 6 & 4-8i \end{pmatrix} \xrightarrow{r_1, r_2 + r_3} \begin{pmatrix} 0 & 3-3i & 6-6i \\ 0 & -4i & -8i \\ -8 & 6 & 4-8i \end{pmatrix} \begin{matrix} \cdot(1+i) \\ \cdot(-\frac{1}{4i}) \\ \div 2 \end{matrix}$$

$$\rightarrow \begin{pmatrix} 0 & \cancel{6} & \cancel{12} \to 0\,0 \\ 0 & 1 & 2 \\ -4 & 3 & 2-4i \end{pmatrix} \xrightarrow[r_3 \div 2]{r_3 - r_2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ -4 & 2 & -4i \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ -2 & 1 & -2i \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ -2 & 0 & -2(1+i) \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1+i \end{pmatrix} \quad \begin{matrix} z = t \\ x = -(1+i)t \\ y = -2t \end{matrix}$$

$$\boxed{\left\{ \begin{pmatrix} -1-i \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1+i \\ -2 \\ 1 \end{pmatrix} \right\} \ \lambda = 1+4i,\ 1-4i \text{ respectively.} \quad \lambda = 1 \pm 4i \text{ geometric multiplicity } 1, \text{ each}}$$

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Q2) a) • The nonlinear streamplot is II. I suspect this because of the tendency for the solutions to follow the "sideways parabola" shape ($\pm\sqrt{x}$ graph-ish shape) as $t \to \infty$. This certainly is not linear and none of the other streamplots have this property.

• The non-diagonalizeable plane system is IV. I believe this because there is only 1 noticeable eigenvector, along the $\begin{bmatrix}1\\4\end{bmatrix}$ line, with a positive eigenvalue tending away from the origin. Since there is only one eigenvector / eigenvalue, it is repeated and hence implying that the corresponding matrix to this system cannot be diagonalized.

b) I : A pair of complex eigenvalues. I choose this because of the spiral coming from the origin as t increases. Complex eigenvalues have this spiral property in phase planes.

III : Two negative real numbers. We can notice two distinct eigenvectors in this phase plane: one along the x-axis and one along the $\begin{bmatrix}1\\2\end{bmatrix}$-ish direction. In both of these cases, the solutions tend to 0 as $t \to \infty$, which implies that there are two negative eigenvalues $e^{\lambda_1 t}, e^{\lambda_2 t}$ where $\lambda_1, \lambda_2 < 0$.

V: One positive and one negative real number.

Like III, we notice two distinct eigenvectors on the phase plane: the x-axis, and approximately $\begin{bmatrix} 1 \\ 10 \end{bmatrix}$.

The solutions located on the x-axis eigenvector are approaching 0 as $t \to \infty$, and the solutions located on the $\begin{bmatrix} 1 \\ 10 \end{bmatrix}$ eigenvector are tending to infinity as $t \to \infty$.

This implies that the eigenvalue associated with the x-axis eigenvector, is negative and likewise, positive with the $\begin{bmatrix} 1 \\ 10 \end{bmatrix}$ eigenvector.

Therefore one positive and one negative real number.

MAT244 PS5

Q3) a) Let $A = \begin{pmatrix} -1 & 4 & 0 \\ -2 & 3 & -2 \\ 2 & -4 & 1 \end{pmatrix}$.

$$\text{char}(A) = (-1-\lambda)(3-\lambda)(1-\lambda) + (4)(-2)(2) + 0 - 0 - (-4)(-2)(-1-\lambda)$$
$$- (1-\lambda)(-2)(4)$$
$$= -3 + 3\lambda^2 + \lambda - \lambda^3 \;\cancel{-16 + 8 + 8\lambda + 8 - 8\lambda} \to 0$$
$$= -\lambda^3 + 3\lambda^2 + \lambda - 3$$

We find $\lambda = 1 \Rightarrow \text{char}(A) = 0$.

$$\begin{array}{r|rrrr} 1 & -1 & 3 & 1 & -3 \\ & & -1 & 2 & 3 \\ \hline & -1 & 2 & 3 & 0 \end{array} \quad \to -(\lambda-1)(\lambda^2-2\lambda-3) \to -(\lambda-1)(\lambda-3)(\lambda+1).$$

$\lambda=1$, $A-I$:
$$\begin{pmatrix} -2 & 4 & 0 \\ -2 & 2 & -2 \\ 2 & -4 & 0 \end{pmatrix} \xrightarrow[r_1+r_3]{r_2+r_3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & -2 \\ 2 & -4 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$$

$y=t$, $z=-t$, $x=2t$

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\} \quad \lambda = 1$$

$\lambda=-1$, $A+I$:
$$\begin{pmatrix} 0 & 4 & 0 \\ \cancel{-2} & \cancel{4} & \cancel{-2} \\ 2 & -4 & 2 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & -4 & 2 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$z=t$, $y=0$, $x=-t$

$$\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \lambda = -1$$

$\lambda=3$, $A-3I$:
$$\begin{pmatrix} -4 & 4 & 0 \\ -2 & 0 & -2 \\ 2 & -4 & -2 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & -1 \\ 1 & -2 & -1 \end{pmatrix} \to \begin{pmatrix} 0 & \overset{1}{\cancel{-1}} & \overset{1}{\cancel{-1}} \\ 0 & \cancel{-2} & \cancel{-2} \\ 1 & -2 & -1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & -2 & -1 \end{pmatrix}$$
$$\to \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}$$

$y=t$
$x=t$
$z=-t$

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\} \quad \lambda = 3.$$

$\lambda=3$ has algebraic, geometric multiplicities 1.

$\lambda=1,-1$ both have algebraic, geometric multiplicities 1

Since the eigenvalues and eigenvectors are all distinct, A is diagonalizable.

b) As from (a), the basis of eigenvectors is

$$\boxed{\left\{\begin{pmatrix}2\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}\right\}}.$$

By diagonalization, $\boxed{P=\begin{pmatrix}2&-1&1\\1&0&1\\-1&1&-1\end{pmatrix}}$ with $\boxed{D=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&3\end{pmatrix}}$.

find P^{-1},

$$\left(\begin{matrix}2&-1&1\\1&0&1\\-1&1&-1\end{matrix}\middle|\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\right) \xrightarrow[r_2+r_3]{r_1+r_3} \left(\begin{matrix}1&0&0\\0&1&0\\-1&1&-1\end{matrix}\middle|\begin{matrix}1&0&1\\0&1&1\\0&0&1\end{matrix}\right)$$

$$\xrightarrow[r_3\cdot(-1)]{r_3+r_1,\ r_3-r_2} \left(\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\middle|\begin{matrix}1&0&1\\0&1&1\\1&-1&1\end{matrix}\right)_{\cdot(-1)} = \begin{pmatrix}1&0&1\\0&1&1\\-1&1&-1\end{pmatrix}.$$

So $\boxed{P^{-1}=\begin{pmatrix}1&0&1\\0&1&1\\-1&1&-1\end{pmatrix}}$. Thus $A=PDP^{-1}$, and $D=P^{-1}AP$.

c) Notice that $e^{At}=\sum_{n=0}^{\infty}\frac{(At)^n}{n!}=\sum_{n=0}^{\infty}\frac{(PDP^{-1})^n t^n}{n!}=\sum_{n=0}^{\infty}\frac{PD^nP^{-1}t^n}{n!}$

$$= P\left(\sum_{n=0}^{\infty}\frac{D^n t^n}{n!}\right)P^{-1} = \boxed{Pe^{Dt}P^{-1}}$$

d) We know $e^{Dt}=\begin{pmatrix}e^t&&\\&e^{-t}&\\&&e^{3t}\end{pmatrix}$, and thus our fundamental matrix is given by $Pe^{Dt}=AP$.

Therefore let $\Psi(t):=Pe^{Dt}$, so $\Psi(0)=P$. Furthermore, our initial conditions are $C=\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$, so $X(0)=P\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$, thus

$$\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}2&-1&1\\1&0&1\\-1&1&-1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} \Rightarrow \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}1&0&1\\0&1&1\\-1&1&-1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}2\\2\\-1\end{pmatrix}.$$

The solution to the IVP is $\boxed{\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}4\\2\\-2\end{pmatrix}e^t+\begin{pmatrix}-2\\0\\2\end{pmatrix}e^{-t}+\begin{pmatrix}-1\\-1\\1\end{pmatrix}e^{3t}}$

MAT244 PS5

Q4) a) We have that $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ x(t) \end{pmatrix} \to \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

Let $D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $\text{char}(D) = (-\lambda)^2 - 1 = \lambda^2 - 1$, so $\lambda = \pm 1$.

$\lambda=1$, $A-I$: $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \to \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ $\lambda = 1$.

$\lambda=-1$, $A+I$: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \to \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$. $\lambda = -1$.

The general solution describing the particle's movement is then

$$\boxed{\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}}$$

b) $x''(t) = y(t)$, $y''(t) = x(t)$. Let $w = x'$, $z = y'$. $\to$ system $x' = w$, $y' = z$, $w' = y$, $z' = x$.

Define $T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. This is the associated matrix (in x, y, z, w order).

We find, easily, that $\text{char}(T) = \lambda^4 - 1$. The roots are then $\lambda = \pm 1, \pm i$.

Eigenvectors;

$\lambda = 1$, $T-I$: $\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ $\lambda = 1$

$\lambda = -1$, $T+I$: $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\}$ $\lambda = -1$.

$\lambda = -i$
$T + iI$

$$\begin{pmatrix} i & 0 & 0 & 1 \\ 0 & i & 1 & 0 \\ 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \end{pmatrix} \begin{matrix} \cdot i \\ \cdot i \\ \\ \\ \end{matrix} \to \begin{pmatrix} -1 & 0 & 0 & i \\ 0 & -1 & i & 0 \\ 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 & i \\ 0 & -1 & i & 0 \\ 0 & 0 & i & i \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} -1 & 0 & 0 & i \\ 0 & -1 & 0 & -i \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \left\{ \begin{pmatrix} i \\ -i \\ -1 \\ 1 \end{pmatrix} \right\} \quad \lambda = -i.$$

Our basis of eigenvectors is then

$$\left\{ \overset{\lambda=1}{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}, \overset{\lambda=-1}{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}}, \overset{\lambda=i}{\begin{pmatrix} -i \\ i \\ -1 \\ 1 \end{pmatrix}}, \overset{\lambda=-i}{\begin{pmatrix} i \\ -i \\ -1 \\ 1 \end{pmatrix}} \right\}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} -i \\ i \\ -1 \\ 1 \end{pmatrix} e^{it} + c_4 \begin{pmatrix} i \\ -i \\ -1 \\ 1 \end{pmatrix} e^{-it}$$

x: $-i(\cos t + i\sin t)$
$= \underline{-i\cos t} + \underline{\sin t}$
y: $i(\cos t + i\sin t)$
$= i\underline{\cos t}\ \underline{-\sin t}$

x: $i(\cos t - i\sin t)$
$= i\underline{\cos t} + \underline{\sin t}$
y: $-i(\cos t - i\sin t)$
$= \underline{-i\cos t}\ \underline{-\sin t}$

Since we are only looking for the x and y solutions,

we have

$$\boxed{\begin{aligned} x(t) &= c_1 e^{t} + c_2 e^{-t} + c_3(\sin t - \cos t) + c_4(\cos t + \sin t) \\ y(t) &= c_1 e^{t} + c_2 e^{-t} + c_3(\cos t - \sin t) + c_4(\cos t + \sin t). \end{aligned}}$$

MAT2444 PS5

Q5) a) Prove that $X(t) = vt^r$ is a solution to the system

$x'(t) = \frac{1}{t} A\, x(t)^{[(1)]}$ if and only if v is an eigenvalue

of A with corresponding eigenvalue r.

proof.

→ • Assume $x(t) = vt^r$ is a solution to (1). Then

$\frac{d}{dt}[vt^r] = vrt^{r-1} = \frac{1}{t} vrt^r$.

However, $\frac{1}{t} vrt^r = \frac{1}{t} Avt^r$, which directly implies that

$Av = rv$, and thus v is a eigenvalue of A with correspondi

eigenvalue r.

← • Conversly, if $Av = rv$ and we consider an equation

of the form $x(t) = vt^r$, then $\frac{1}{t} Avt^r = vrt^{r-1} =$

$\frac{d}{dt}[vt^r] = x'(t)$, and thus $x(t)$ is a solution to (1).

$\square$

b) • The wronskian $W[x_1, \cdots, x_n] = \det \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1' & x_2' & \cdots & x_n' \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$ is 0 if

and only if any column is linearly dependent. It suffices to show

that if $x_1, \cdots, x_n$ are linearly independent, they form a fundamental

set of solutions. Notice that for any arbitrary $x_i(t)$,

the eigenvector v_i is just scaled by t^{r_i}, hence not affectin

linear dependence with the other eigenvectors.

Thus since the basis of eigenvectors $v_1, \cdots, v_n$ are linearly

independent, then so are $x_1, \cdots, x_n$.

• Therefore $X_1, \cdots, X_n$ form a fundamental set of solutions.

$\square$

c) For $t X'(t) = A X(t)$, where $A = \begin{pmatrix} -1 & 4 & 0 \\ 2 & 3 & -2 \\ 2 & -4 & 1 \end{pmatrix}$ from Q3.

From the previous Lemma's presented in (Q5 a, b), we can form a solution:

Recall from Q3: $V_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ $\lambda_1 = 1$

$V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\lambda_2 = -1$

$V_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ $\lambda_3 = 3$

And thus our general solution is

$$\boxed{X(t) = C_1 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} t + C_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} t^{-1} + C_3 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} t^3}$$