PHY250 PS5 1006940802 - Jace Alloway April 10th 2022

I have read and I understand the homework set policy in this course.

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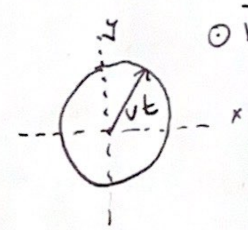
Q1

To find the emf, we can begin by finding the magnetic flux through the ring as a function of time.



$\vec{B} = B_0(1+Kt)\hat{z}$, and $\Phi = \iint \vec{B}\cdot d\vec{a}$.

The infinitesimal area element $d\vec{a}$ of the circle is $d\vec{a} = r\,dr\,d\theta\,\hat{z}$, orienting the circle in the positive $\hat{z}$ direction with the magnetic field.

Here, $0 \leq \theta \leq 2\pi$ and $0 \leq r' \leq r = vt$. The flux is then

$$\Phi = \int_0^{2\pi}\int_0^{vt} (B_0(1+Kt)\hat{z}) \cdot (r\,dr\,d\theta\,\hat{z})$$

$$= 2\pi B_0 \int_0^{vt} (1+Kt)\, r\, dr$$

$$= \pi B_0 (1+Kt) v^2 t^2.$$

The emf is related to the flux through the loop by $\mathcal{E} = -\dot{\Phi}$. Taking the time derivative, we have that

$$\mathcal{E} = -\dot{\Phi} = -\pi B_0 \frac{d}{dt}\left[(1+Kt)v^2t^2\right]$$

$$= -\pi B_0\left[2v^2t(1+Kt) + Kv^2t^2\right]$$

which simplifies to

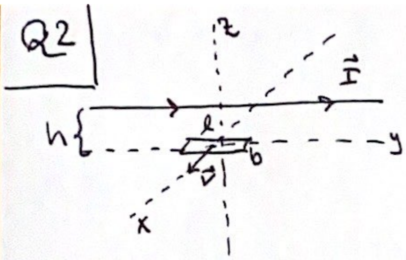
$$\boxed{\mathcal{E} = -B_0 v^2 \pi (3Kt^2 + 2t)}.$$

Faraday stated that an emf is created when either $\vec{B}$ changes, or the current is moved.

Note that when $K=0$, $\vec{B} = \text{const}$ and $\mathcal{E} = -B_0 v^2 \pi \cdot 2t$, which is the emf created just by heating the loop. Now suppose $r = \text{constant}$, and $\vec{B}$ varies. Then $\mathcal{E} = -\dot{\Phi} = -\frac{d}{dt}(\pi r^2 B_0(1+Kt)) = -K\pi r^2 B_0 \neq 0$.

In either case, $\mathcal{E} \neq 0$. Therefore the solution $\mathcal{E}$ is a concatenation of those two currents, and thus $\mathcal{E}$ is from $\vec{B}$ varying and the loop radius changing.

Q2



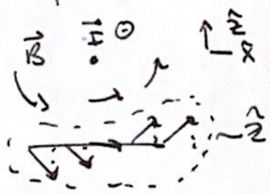
To find the emf induced on the hoop, we can consider the magnetic flux through the hoop at a fixed time t.

Since the loop slides with constant velocity $\vec{v} = v\hat{x}$, then the position of the hoop is given by $x(t) = vt$ in the $\hat{x}$ direction. If $t = t_0$ = fixed, the hoop will be at position x_0.

By ~~Faraday's Law~~ Ampere's Law, the magnetic field from the wire is

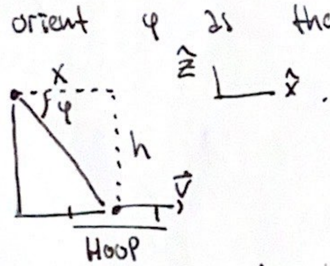
$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{encl} \implies \vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\varphi},$$ where the direction is given by the right-hand rule.

Since the loop is described in cartesian coordinates, and $\vec{B}$ in cylindrical coordinates, it would be best to transform $\vec{B}$ also into cartesian coordinates. How do we orient φ? At $x=0$, we expect the flux of $\vec{B}$ from the wire through the hoop to be zero by symmetry:



$\hat{z}$ components cancel in flux, so $\Phi(x=0) = 0$.

We can orient φ as then



The unit vector is $\hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{z}$, and $\cos\varphi = \frac{x}{\sqrt{x^2+h^2}}$, $\sin\varphi = \frac{h}{\sqrt{x^2+h^2}}$.

Orienting the surface bounded by the hoop in the negative $\hat{z}$ direction gives

$$d\vec{a} = -dx\,dy\,\hat{z},$$ where $x_0 - \frac{b}{2} \le x \le x_0 + \frac{b}{2}$ and $-\frac{\ell}{2} \le y \le \frac{\ell}{2}$.

The flux is then

$$\Phi = \iint \vec{B}\cdot d\vec{a} = \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}}\int_{x_0-\frac{b}{2}}^{x_0+\frac{b}{2}} \frac{\mu_0 I}{2\pi\sqrt{x^2+h^2}}\left(-\frac{h\hat{x}}{\sqrt{x^2+h^2}} + \frac{x\hat{z}}{\sqrt{x^2+h^2}}\right)\cdot dx\,dy\,\hat{z}(-1)$$

$$= -\frac{\mu_0 I \ell}{2\pi}\int_{x_0-\frac{b}{2}}^{x_0+\frac{b}{2}} \frac{x}{\sqrt{x^2+h^2}}\,dx.$$

To find the emf given by $\mathcal{E} = -\frac{d\Phi}{dt}$, we can proceed by differentiation Φ under the integral sign:

$$\mathcal{E} = -\dot{\Phi} = \frac{\mu_0 I \ell}{2\pi} \frac{d}{dt} \int_{vt-\frac{b}{2}}^{vt+\frac{b}{2}} \frac{x}{x^2+h^2}\, dx.$$

$$= \frac{\mu_0 I \ell}{2\pi} \left[\frac{vt+\frac{b}{2}}{(vt+\frac{b}{2})^2+h^2} - \frac{vt-\frac{b}{2}}{(vt-\frac{b}{2})^2+h^2} + \cancel{\int_{vt-\frac{b}{2}}^{vt+\frac{b}{2}} \frac{\partial}{\partial t} \frac{x}{x^2+h^2}\, dx}^{\,0} \right]$$

$$= \frac{\mu_0 I \ell}{2\pi} \left[\frac{x+\frac{b}{2}}{(x+\frac{b}{2})^2+h^2} - \frac{x-\frac{b}{2}}{(x-\frac{b}{2})^2+h^2} \right].$$

We can now apply the approximation that $|x| \gg b$, or $1 \gg \frac{b}{|x|}$. For very small b, we have by Taylor expanding that $f(x+b) \simeq f(x) + b f'(x) + \text{HOT} \dots$.
With our function being given by $f(x) = \frac{x}{x^2+h^2}$, then $f(x \pm \frac{b}{2}) = \frac{x \pm \frac{b}{2}}{(x \pm \frac{b}{2})^2 + h^2}$, which is the expression given above. $f'(x)$ is given by $f'(x) = \frac{h^2-x^2}{(x^2+h^2)^2}$. Therefore our approximation becomes

$$\mathcal{E} = \frac{\mu_0 I \ell}{2\pi} \left[\frac{x}{x^2+h^2} + \frac{b}{2} \cdot \frac{h^2-x^2}{(x^2+h^2)^2} - \frac{x}{x^2+h^2} + \frac{b}{2} \cdot \frac{h^2-x^2}{(x^2+h^2)^2} \right]$$

$$= \frac{\mu_0 I \ell b}{4\pi} \left[\frac{2(h^2-x^2)}{(x^2+h^2)^2} \right]$$

Therefore the emf is

$$\boxed{\mathcal{E} = \frac{\mu_0 I \ell b}{2\pi} \left(\frac{h^2-x^2}{(x^2+h^2)^2} \right)}.$$

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b) To find the maximum and minimum values of $\mathcal{E}$, we can set the first derivative to zero: $\frac{d\mathcal{E}}{dx} = 0$. The derivative (by Wolfram alpha) is

$$\frac{d\mathcal{E}}{dx} = \frac{\mu_0 I \ell b}{2\pi} \left[\frac{2x(x^2-3h^2)}{(x^2+h^2)^3} \right] = 0,$$

which just implies that either $x=0$ or $x^2 = 3h^2$. The maximum and minimum values of $\mathcal{E}$ then occur when $\boxed{x=0 \text{ and when } x = \pm\sqrt{3}\, h}$.

Q3

For a solenoid of n turns per unit length and infinite length, the magnetic field is given by

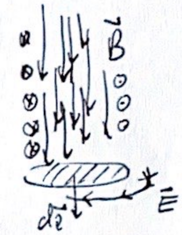
$$\vec{B} = \mu_0 n I \hat{z} \quad (\text{~~outside~~ inside}) \quad \text{and} \quad \vec{B} = \vec{0} \quad (\text{~~inside~~ outside})$$

With $I(t) = ct$, then $\vec{B}(t) = \begin{cases} \mu_0 n c t & (r<R) \\ 0 & (r>R) \end{cases}$.

Faraday's Law gives the relation

$$\oint \vec{E} \cdot d\vec{\ell} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a} .$$

If we orient the surface as a circle in the middle of the solenoid then the electric field is in the $\hat{\varphi}$ direction so:



<u>$r<R$:</u> $d\vec{a} = r'dr'\,d\theta\,\hat{z}$, $\vec{B}(t) = \mu_0 n c$

So $\oint \vec{E}\cdot d\vec{\ell} = 2\pi r E$

and $-\int_0^{2\pi}\int_0^{r} \mu_0 n c\, r'dr'd\theta = -\pi \mu_0 n c r^2 = \iint \dot{\vec{B}} \cdot d\vec{a}$

$\Longrightarrow$ $2\pi r E = -\pi \mu_0 n c r^2$, so $\vec{E} = -\frac{\mu_0 n c}{2} r \hat{\varphi}$.

<u>$r>R$:</u> $\vec{B} = 0$ but $\vec{B}$ still contributes from R:

$$-\iint \dot{\vec{B}} \cdot d\vec{a} = -\int_0^{2\pi}\left[\int_0^{R} \dot{\vec{B}}\cdot d\vec{a} + \cancel{\int_R^{r} \dot{\vec{B}}\cdot d\vec{a}}\right]$$

$$= -\int_0^{2\pi}\int_0^{R} \mu_0 n c \cdot r dr d\theta$$

$$= -\pi \mu_0 n c R^2.$$ Again with $\oint \vec{E}\cdot d\vec{\ell} = 2\pi r E$ for $r>R$,

then $\vec{E} = -\frac{\mu_0 n c R^2}{2r}\hat{\varphi}$.

Therefore the electric field of the solenoid is

$$\boxed{\vec{E} = \begin{cases} -\frac{\mu_0 n c}{2} r \,\hat{\varphi} & [r<R] \\ -\frac{\mu_0 n c R^2}{2r}\,\hat{\varphi} & [r>R]. \end{cases}}$$

It now follows to check if the differential form of Faraday's law holds:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

In cylindrical coordinates,

$$\vec{\nabla} \times \vec{E} = \left(\frac{1}{r} \cancelto{0}{\frac{\partial E_z}{\partial \varphi}} - \cancelto{0}{\frac{\partial E_\varphi}{\partial z}} \right) \hat{r} + \left(\cancelto{0}{\frac{\partial E_r}{\partial z}} - \cancelto{0}{\frac{\partial E_z}{\partial r}} \right) \hat{\varphi} + \frac{1}{r} \left(\frac{\partial (r E_\varphi)}{\partial r} - \cancelto{0}{\frac{\partial E_r}{\partial \varphi}} \right) \hat{z}$$

since $\vec{E} = E_\varphi \hat{\varphi}$.

For $\underline{r < R:}$

$$\vec{E} = -\frac{\mu_0 n c}{2} r \hat{\varphi}, \quad \text{so} \quad \vec{\nabla} \times \vec{E} = -\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\mu_0 n c r^2}{2} \right) \hat{z}$$

$$= -\frac{1}{r} \mu_0 n c r \hat{z}$$

$$= -\mu_0 n c \frac{\cancel{r}}{\cancel{r}} \hat{z}$$

$$\underline{= -\mu_0 n c \hat{z}}$$

but since $\vec{B} = \mu_0 n c t \hat{z}$ then $-\frac{\partial B}{\partial t} \underline{= -\mu_0 n c \hat{z}}$. ✓ $r < R$ holds.

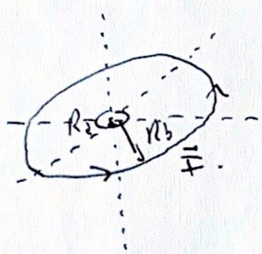
And for $\underline{r > R;}$

$$\vec{\nabla} \times \vec{E} = \frac{1}{r} \frac{\partial}{\partial r} \left(-\frac{\mu_0 n c R^2}{2r} r \right) \hat{z} = \frac{1}{r} (0) \hat{z}$$

$\rightarrow \vec{B}(r > R) = \vec{0}$, and thus ✓ $r > R$ holds.

Therefore the differential and integral forms hold.

Q4) In this problem, only the outer ring is having a current run through it.
In the assumption that $R_a << R_b$, it suffices to find the magnetic field at the center of the hoops, then calculate the flux of $\vec{B}_b$ through hoop 'a' to find the mutual inductance.



The Biot-Savart law gives

$$\vec{B}(r) = \frac{\mu_0 I}{4\pi} \int \frac{dl' \times \hat{\mathscr{r}}}{\mathscr{r}^2}.$$

From the center, $\vec{\mathscr{r}} \equiv -R_b \hat{r}$, and $dl' = R_b \, d\varphi \, \hat{\varphi}$.

Then $\vec{B}(0) = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{R_b \, d\varphi}{R_b^2} \hat{z}$ $\quad (\hat{z} \equiv \hat{\varphi} \times \hat{r})$

$$= \frac{\mu_0 I}{2R_b} \hat{z}$$

The flux through ring 'a' is then $\Phi_1 = \int \vec{B}_b(0) \cdot d\vec{a}$, with $d\vec{a} = r\,dr\,d\varphi \, \hat{z}$,

or $\Phi_1 = \frac{\mu_0 I}{2R_b} \int_0^{2\pi} \int_0^{R_a} r\,dr\,d\varphi = \frac{\mu_0 \pi R_a^2 I}{2R_b}$.

From the mutual inductance relation

$$\Phi_1 = M_{12} I_2,$$

we have that $\Phi_a = M_{ab} I_b$ or that $\frac{\mu_0 \pi R_a^2 I}{2R_b} = M_{ab} I$,

and therefore the mutual inductance is

$$\boxed{M_{ab} = \frac{\mu_0 \pi R_a^2}{2R_b}}$$

Q5 | a)

To show $\vec{B}=\vec{0}$, we can examine the Maxwell-Faraday equation

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}.$$

Since $\vec{E} = E_1(x,t)\,\hat{x}$, then the curl of $\vec{E}$ is

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_1 & 0 & 0 \end{vmatrix} = 0\hat{x} + \partial_z E_1\,\hat{y} - \partial_y E_1\,\hat{z}$$

$$= 0,$$

which follows from $\vec{E}$ being independent of y and z.

This implies that $\frac{\partial\vec{B}}{\partial t} = 0$, or that there is no time evolution in the magnetic field.

With the electromagnetic component being stationary in time, then the $\vec{B}$ field for the wave is zero.

$$\boxed{\vec{B}=\vec{0}.}$$

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b) The current density $\vec{J}$ is given by the product between the number of charges of charge q moving at velocity $\vec{v}$.

That is, $\vec{J} = e\,n_p\vec{v}_p + (-e)\,n_e\vec{v}_e.$

This follows from electrons and protons each sharing the same magnitude of charge e. With

$$n_p = n_0 + n_{1p}\,,\quad n_e = n_0 + n_{1e}\,,\quad \vec{v}_p = v_{1p}\,\hat{x} \quad\text{and}\quad \vec{v}_e = v_{1e}\,\hat{x},$$

the current density is

$$\boxed{\vec{J} = e(n_0 + n_{1p})(\vec{v}_{1p}) - e(n_0 + n_{1e})(\vec{v}_{1e})}$$

In the linear approximation $n_0 \gg n_{1p}, n_{1e}$, it follows that

$$\vec{J} = e n_0\Big(1 + \underbrace{\frac{n_{1p}}{n_0}}_{\approx 0}\Big)\vec{v}_{1p} - e n_0\Big(1 + \underbrace{\frac{n_{1e}}{n_0}}_{\approx 0}\Big)\vec{v}_{1e}$$

which then approximates $\vec{J}$ to

$$\vec{J} = en_0 (\vec{v}_{1p} - \vec{v}_{1e}).$$

Now, the curl of $\vec{B}$ gives

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t},$$

but since $\vec{B} = \vec{0}$, then

$$-\vec{J} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}.$$

Moreover, since both $\vec{J}$ and $\vec{E}$ are in the $\hat{x}$ direction, we just need to consider the components. We have

$$-en_0 (v_{1p} - v_{1e}) = \epsilon_0 \frac{\partial E_1}{\partial t},$$ which then implies that

$$\boxed{\frac{\partial E_1}{\partial t} = -\frac{en_0}{\epsilon_0}(v_{1p} - v_{1e})},$$ as desired.

(c) For this part, I will invoke the Lorentz Force law, since $\vec{B} = \vec{0}$, we have that

$$m\vec{a} = q\vec{E} \implies m \frac{dv_x}{dt} = qE_x.$$

The electric field is given as before, $E_x = E_1(x,t)$.

For electrons, $m = m_e$ and $q = -e$. Then

$$\boxed{\frac{dv_{1e}}{dt} = -\frac{e}{m_e} E_1(x,t)}$$

Similarly for protons, $m = m_p$ and $q = +e$, so

$$\boxed{\frac{dv_{1p}}{dt} = \frac{e}{m_p} E_1(x,t).}$$

Note that this expression is only valid for the linear approximation that

$v_e = v_{1e}$ and $v_p = v_{1p}$.

These second order ODE's in time describe the motion of each the protons and electrons.

Since the derivatives of $\cos x$, $\sin x$ cycle, we have that

$$\frac{dV_{1p}}{dt} = -\frac{m_e}{m_p}\frac{dV_{1e}}{dt},$$

and the motions are oscillatory due to $E_1(x,t) = E_0 \cos(\omega t + kx)$.

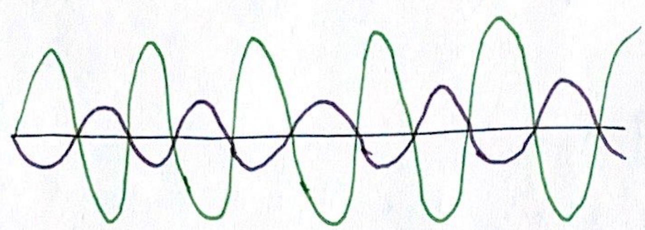
Furthermore,

$$r_{1e} = \frac{e}{m_e}\frac{1}{\omega^2} E_0 \cos(\omega t + kx) \qquad r_{1p} = -\frac{e}{m_p}\frac{1}{\omega^2} E_0 \cos(\omega t + kx)$$

$$\dot{r}_{1e} = V_{1e} = -\frac{e}{m_e}\frac{1}{\omega} E_0 \sin(\omega t + kx) \qquad \dot{r}_{1p} = V_{1p} = \frac{e}{m_p}\frac{1}{\omega} E_0 \sin(\omega t + kx)$$

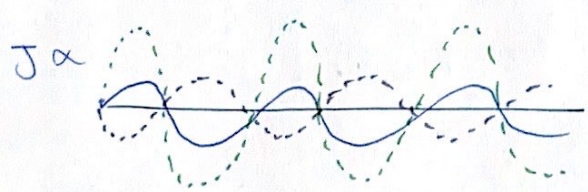
$$\ddot{r}_{1e} = \dot{V}_{1e} = -\frac{e}{m_e} E_1(x,t) \qquad \ddot{r}_{1p} = \dot{V}_{1p} = \frac{e}{m_p} E(x,t).$$

The motion of the two particles is sinusoidal and opposite:



Taking the difference of the two $V_{1p} - V_{e1}$, which is proportional to the current density, is also oscillatory, since $m_e \neq m_p$.

Therefore the current density oscillates back and forth in ~~magnitude~~ ~~and~~ sign:



d) To find the dispersion relation, we can utilize the expressions from the previous parts (b) and (c).

From (b), we have that

$$\frac{\partial E_1}{\partial t} = -\frac{e n_0}{\varepsilon_0}(V_{1p} - V_{1e}),$$

and from (c), we have then

$$\frac{\partial^2 E}{\partial t^2} = -\frac{e n_0}{\varepsilon_0}\left(\frac{\partial V_{1p}}{\partial t} - \frac{\partial V_{1e}}{\partial t}\right)$$

$$= -\frac{e n_0}{\varepsilon_0}\left(\frac{e}{m_p} - \left(-\frac{e}{m_e}\right)\right) E_1(x,t)$$

$$= -\frac{e^2 n_0}{\varepsilon_0}\left(\frac{1}{m_p} + \frac{1}{m_e}\right) E_1(x,t)$$

$$= -\frac{e^2 n_0}{\varepsilon_0}\left(\frac{m_e + m_p}{m_p m_e}\right) E_1(x,t) \equiv -\frac{e^2 n_0}{\varepsilon_0} \cdot \frac{1}{\mu} E_1(x,t),$$

by letting the reduced mass be $\mu = \frac{m_p m_e}{m_e + m_p}$.

Therefore

$$\ddot{E}_1 = -\frac{e^2 n_0}{\varepsilon_0 \mu} E_1.$$

Since $E_1(x,t) = E_0 \cos(\omega t + kx)$, then $\dot{E}_1 = -\omega E_0 \sin(\omega t + kx)$ and $\ddot{E}_2 = -\omega^2 E_0 \cos(\omega t + kx) \equiv -\omega^2 E_1$, so utilizing the previous expression gives

$$\ddot{E}_1 = -\omega^2 E_1 = -\frac{e^2 n_0}{\varepsilon_0 \mu} E_1,$$

which implies that $\omega^2 = \frac{e^2 n_0}{\varepsilon_0 \mu}$.

The dispersion relation is then $\boxed{\omega = \sqrt{\frac{e^2 n_0}{\varepsilon_0 \mu}}}$.

e) To find the time average EM energy density, we can invoke poynting's theorem, which states that the energy stored in electromagnetic fields per unit volume is given by

$$u = \frac{1}{2}\left[\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right].$$

However, since $\vec{B} = \vec{0}$, then

$$u = \frac{\epsilon_0}{2}\vec{E}^2.$$

The time average over the period is given by $\bar{u} = \frac{1}{T}\int_0^T u\,dt$, so

$$\bar{u} = \frac{1}{T}\int_0^T \epsilon_0 E_0^2 \cos^2(\omega t + kx)\,dt$$

$$= \frac{1}{T}\int_0^T \epsilon_0 E_0^2 \cos^2(\omega t)\,dt$$

(since shifting $\cos(\omega t + kx) \to \cos(\omega t)$ does not affect the period and hence the period average)

$$= \frac{\epsilon_0 E_0^2}{T}\int_0^T \cos^2(\omega t)\,dt$$

$$= \frac{\epsilon_0 E_0^2}{T}\left[\frac{\sin(2\omega t)}{4\omega} + \frac{t}{2}\right]_0^T$$

(since $\sin(0) = \sin\left(2 \cdot \frac{2\pi}{T} \cdot T\right) = \sin(4\pi) = 0$)

$$= \frac{\epsilon_0 E_0^2}{T} \cdot \left(\frac{T}{2} + 0\right)$$

$$= \frac{\epsilon_0 E_0^2}{2}.$$

And therefore the time average EM energy density is

$$\boxed{\bar{u} = \frac{\epsilon_0 E_0^2}{2}.}$$