PS2

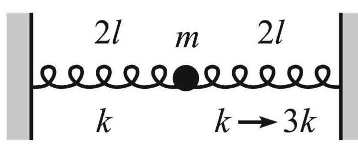
My score

90.6% (14.5/16)

Q1

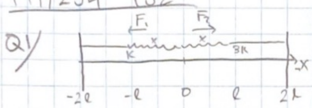
4 / 4

1. Two springs each have spring constant k and equilibrium length l. They are both stretched a distance l and attached to a mass m and two walls, as shown in figure below. At a given instant, the right spring constant is somehow magically changed to 3k (the relaxed length remains l). What is the resulting x(t)? Take the initial position to be x = 0.



PHY254 PS2

Q1)



$\vec{F}_1 = -k(\ell + x)$ [LEFT]

$\vec{F}_2 = -3k(\ell - x)$ [RIGHT]

Then $F = ma = -k(\ell+x) + 3k(\ell - x)$

$\Rightarrow m\ddot{x} = -k\ell - kx + 3k\ell - 3kx$

$= 2k\ell - 4kx$

- Find the homogeneous solution:

$\ddot{x} + 4\frac{k}{m}x = 0$ Guess $y = Ae^{rt}$

$r^2 + 4\frac{k}{m} = 0 \Rightarrow r^2 = -4\frac{k}{m}$

$\Rightarrow r = \pm i 2\sqrt{\frac{k}{m}}$

Thus our homogeneous solution is $x_h(t) = A_1 e^{i \cdot 2\sqrt{k/m}\, t} + A_2 e^{-i2\sqrt{k/m}\, t}$

- Find a particular solution:

$\ddot{x} + 4\frac{k}{m}x = 2\frac{k}{m}\ell$ If $x_p(t)$ is a constant, $\ddot{x}_p = 0$.

$\Rightarrow \frac{4}{m}x = \frac{2\ell}{m} \Rightarrow x_p = \frac{1}{2}\ell$.

Our general solution is then

$x = x_h + x_p = A_1 e^{2i\sqrt{k/m}\, t} + A_2 e^{-2i\sqrt{k/m}\, t} + \frac{1}{2}\ell$.

Q2

3 / 4

2. (a) Two vibrations are given by:
$x_1 = A\cos(\omega_1 t + \phi_1)$ and $x_2 = A\cos(\omega_2 t + \phi_2)$.

Find an expression for their superposed motion in the form

$$a\cos(bt + c)\cos(dt + e)$$

and determine the constants a through e in terms of the given parameters. Use complex notation in your derivation, as in the class notes.

(b) Consider the vibrations $x_1 = 3\cos(3t + \pi/5)$ and $x_2 = 4\cos(4t + \pi/8)$.

What is the expected period for $x_1 + x_2$?

(c) **Complex number and phasor practice:** Express the following in the form $z = \mathrm{Re}[Ae^{i(\omega t+\alpha)}]$ and sketch them by hand or plot them in pylab in the complex plane at $t = \pi/\omega$ (i.e. at one half period):

$z = 5\sin\omega t + 6\cos\omega t$;
$z = \sin(\omega t + \pi/3) - \cos\omega t$;
$z = \sin\omega t + 2\cos(\omega t + \pi/6) + \cos\omega t$.

note: just sketch by hand is fine
For the sketch, you should sketch $Ae^{i(\omega t+\alpha)}$ (not z, since z is real) in the complex plane at $t = \pi/\omega$

PHY254 PS2

Q2) a) $x_1 = A\cos(\omega_1 t + \phi_1)$, $x_2 = A\cos(\omega_2 t + \phi_2)$.

$\rightarrow$ Same amplitude!

$x_1 = \mathrm{Re}\{Ae^{i(\omega_1 t + \phi_1)}\}$, $x_2 = \mathrm{Re}\{Ae^{i(\omega_2 t + \phi_2)}\}$.

Then $x_1 + x_2 = \mathrm{Re}\{Ae^{i(\omega_1 t + \phi_1)} + Ae^{i(\omega_2 t + \phi_2)}\}$.

Let $\omega_1 t + \phi_1 = \alpha + \beta$, $\omega_2 t + \phi_2 = \alpha - \beta$.

$\Rightarrow \alpha = \omega_1 t + \phi_1 - \beta$

$\omega_2 t + \phi_2 = \omega_1 t + \phi_1 - 2\beta \quad \Rightarrow \beta = -\frac{1}{2}\left[(\omega_2 - \omega_1)t + (\phi_2 - \phi_1)\right]$

$\Rightarrow \alpha = \frac{1}{2}\left[(\omega_2 + \omega_1)t + (\phi_2 + \phi_1)\right]$.

Let $\omega_2 - \omega_1 = \Delta\omega$, $\phi_2 - \phi_1 = \Delta\phi$.

And $\omega_2 + \omega_1 = \delta\omega$, $\phi_2 + \phi_1 = \delta\phi$.

$\Rightarrow x_1 + x_2 = \mathrm{Re}\{Ae^{i(\alpha+\beta)} + Ae^{i(\alpha-\beta)}\} = \mathrm{Re}\{Ae^{i\alpha}(e^{i\beta} + e^{-i\beta})\}$

However $e^{i\beta} + e^{-i\beta} = \cos(\beta) + i\sin(\beta) + \cos(\beta) - i\sin(\beta)$

$= 2\cos(\beta)$.

$= \mathrm{Re}\left\{Ae^{\frac{i}{2}\delta\omega t} e^{\frac{i}{2}\delta\phi}\left[\cdot 2\cos(-\frac{1}{2}\Delta\omega t - \frac{1}{2}\Delta\phi)\right]\right\}$

$= \mathrm{Re}\left\{A\left(\cos(\frac{\delta\omega}{2}t) + i\sin(\frac{\delta\omega}{2}t)\right)\left(\cos(\frac{\delta\phi}{2}) + i\sin(\frac{\delta\phi}{2})\right)\cdots\right.$

$= \mathrm{Re}\left\{2A\underbrace{\left(\cos(\frac{\delta\omega}{2}t)\cos(\frac{\delta\phi}{2}) - \sin(\frac{\delta\omega}{2}t)\sin(\frac{\delta\phi}{2}\right)}_{= \cos\left(\frac{\delta\omega}{2}t + \frac{\delta\phi}{2}\right)} + [\text{imaginary terms}]\cdots\right.$

$= 2A\cos\left(\frac{\delta\omega}{2}t + \frac{\delta\phi}{2}\right)\cos\left(\frac{-\Delta\omega}{2}t + \frac{-\Delta\phi}{2}\right)$

Thus

$$\boxed{x_1 + x_2 = 2A\cos\left(\frac{\omega_1 + \omega_2}{2}t + \frac{\phi_1 + \phi_2}{2}\right)\cos\left(\frac{\omega_1 - \omega_2}{2}t + \frac{\phi_1 - \phi_2}{2}\right)}$$

b) Consider $x_1 = 3\cos(3t + \frac{\pi}{5})$, $x_2 = 4\cos(4t + \frac{\pi}{8})$

Expected period $T_{ex} = \frac{2\pi}{\omega_{ex}}$.

Notice that $\omega_1 = 3$, $\omega_2 = 4$

$\Rightarrow \frac{\omega_1}{\omega_2} = \frac{3}{4} = \frac{m}{n} = r$,

so ω_1 & ω_2 are commensurate.

This implies that $x_1 + x_2$ has a periodic superposition.

$mT_1 = 3 \cdot \frac{2\pi}{3} = 2\pi$, and

$nT_2 = 4 \cdot \frac{2\pi}{4} = 2\pi$,

Therefore $\boxed{T_{ex} = 2\pi}$.

c) Find and plot the phasor diagrams in the form $z = \mathrm{Re}\{Ae^{i(\omega t + \phi)}\}$.

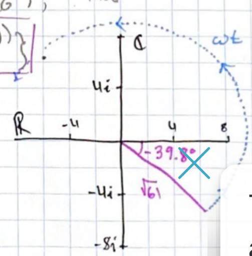
i) $z_1 = 5\sin\omega t + 6\cos\omega t$.

$= \mathrm{Re}\{5\cdot(-i)e^{i(\omega t)} + 6e^{i(\omega t)}\} = \mathrm{Re}\{(6-5i)e^{i\omega t}\}$.

Express $6-5i$ as some $e^{i\phi}$. $r = \sqrt{6^2 + (-5)^2} = \sqrt{61}$,

and $\phi = \arctan(-\frac{5}{6}) = -\arctan(\frac{5}{6})$; thus

$\boxed{z_1 = \mathrm{Re}\{\sqrt{61}\, e^{i(\omega t - \arctan(5/6))}\}}$.



This is not at t=pi/omega

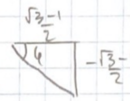
ii) $z_2 = \sin(\omega t + \frac{\pi}{3}) - \cos\omega t$

$= \mathrm{Re}\left\{-i\, e^{i(\omega t + \pi/3)}\right\} - \mathrm{Re}\left\{e^{i\omega t}\right\}$

$= \mathrm{Re}\left\{(-1-i)\, e^{i\omega t}\, \underbrace{e^{i\pi/3}}_{= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}}\right\}$

$= \mathrm{Re}\left\{(-1-i)\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) e^{i\omega t}\right\}$

$= \mathrm{Re}\left\{\underbrace{\left(\frac{\sqrt{3}-1}{2} - i\frac{\sqrt{3}+1}{2}\right)} e^{i\omega t}\right\}$



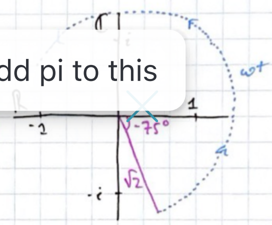
$r = \sqrt{\left(\frac{\sqrt{3}-1}{2}\right)^2 + \left(\frac{\sqrt{3}+1}{2}\right)^2} = \sqrt{2}$

$\varphi = \arctan\left(-\frac{\frac{\sqrt{3}+1}{2}}{\frac{\sqrt{3}-1}{2}}\right)$

$= -\frac{5\pi}{12}$

Need to add pi to this

Thus $\boxed{z_2 = \mathrm{Re}\left\{\sqrt{2}\, e^{i(\omega t - \frac{5\pi}{12})}\right\}}$



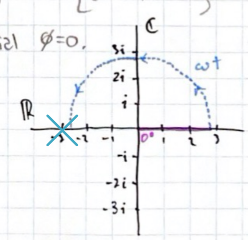
iii) $z_3 = \sin\omega t + 2\cos(\omega t + \frac{\pi}{6}) + \cos\omega t$

$= \mathrm{Re}\left\{-i e^{i\omega t}\right\} + \mathrm{Re}\left\{2e^{i(\omega t + \frac{\pi}{6})}\right\} + \mathrm{Re}\left\{e^{i\omega t}\right\}$

$= \mathrm{Re}\left\{(-i + 2e^{i\pi/6} + 1)\, e^{i\omega t}\right\}$

$= \mathrm{Re}\left\{(-i + 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) + 1)e^{i\omega t}\right\} = \mathrm{Re}\left\{(\sqrt{3}+1)\, e^{i\omega t}\right\}$

Thus $\boxed{z_3 = \mathrm{Re}\left\{(\sqrt{3}+1)\, e^{i\omega t}\right\}}$ initial $\phi = 0$.



Q3

4 / 4

3. A mass m connected to an ideal spring slides frictionlessly on a surface. It is connected to a "dashpot" damper that produces a resistive force proportional to the velocity v of the mass.

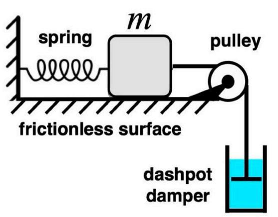


Figure 1: Damped oscillator

We did some experiments on this device and made the following observations:

- If the block is pushed horizontally to the left with a static force equal to mg, the spring compresses a distance h.
- If the block is moved at a steady speed $U = 3\sqrt{gh}$, the dashpot damper produces a resistive force of magnitude mg.

Now the system is displaced and released from rest, setting it into oscillation.

(a) Find the angular frequency of the underdamped oscillations of the system, expressed as a multiple of $\sqrt{g/h}$.

(b) After what time, expressed as a multiple of $\sqrt{h/g}$, is the amplitude of the underdamped motion decreased by a factor of e from its initial displacement?

PHY254 PS2

Q3) Forces: $F = -bv$ damping force

$F = -kx$ restoring spring force.

$F = mg = -k(-h)$, $\Rightarrow k = \frac{mg}{h}$. Similarly,

$F = mg = -b \cdot 3\sqrt{gh}$, $\Rightarrow b = \frac{m}{3}\sqrt{\frac{g}{h}}$.

Our ODE is then $F = F_{sp} + F_{damp}$, so

$$ma = m\ddot{x} = -\frac{mg}{h}x - \frac{m}{3}\sqrt{\frac{g}{h}}\,\dot{x}.$$

This implies that $\ddot{x} + \frac{g}{h}x + \frac{1}{3}\sqrt{\frac{g}{h}}\,\dot{x} = 0.$

a) Find angular frequency. We want a solution of the form $x = e^{irt} \cdot A$. Let $p = \sqrt{\frac{g}{h}}$ for simplicity. Guess $x_h = Ae^{rt}$. Then

$r^2 + \frac{1}{3}pr + p^2 = 0.$ $\Rightarrow 3r^2 + pr + 3p^2 = 0.$

Quadratic formula: $r = \frac{-p \pm \sqrt{p^2 - 4(3)(3p^2)}}{6} = \frac{-p \pm p\sqrt{1-36}}{6}$

$$\Rightarrow r = -\frac{p}{6} \pm i\frac{p\sqrt{35}}{6}.$$

Our general solution is then

$$x = Ae^{-\frac{pt}{6}}e^{i\frac{p\sqrt{35}}{6}t} + Be^{-\frac{pt}{6}}e^{-i\frac{p\sqrt{35}}{6}t}.$$

Since the angular frequency ω is given in the complex solution $e^{i\omega t}$, this implies that $\omega = p \cdot \frac{\sqrt{35}}{6}$.

Thus $\boxed{\omega = \frac{\sqrt{35}}{6} \cdot \sqrt{\frac{g}{h}}}$

b) Time for amplitude to decay a e^{-1} factor:

In the solution to the ODE, the amplitude term is decayed by the term $e^{-\frac{pt}{6}} = e^{-\sqrt{\frac{g}{h}}\cdot\frac{t}{6}}$.

$\Rightarrow$ Solve for t in $e^{-\sqrt{\frac{g}{h}}\cdot\frac{t}{6}} = e^{-1}$,

which implies that

$$\sqrt{\frac{g}{h}}\cdot\frac{T}{6} = 1, \quad \text{or} \quad \boxed{T = 6\cdot\sqrt{\frac{h}{g}}},$$

as desired.

Q4

3.5 / 4

4. Consider a particle of mass m, free to move in one dimension x, which is subjected to a drag force $-bv = -b\dot{x}$ but *no* spring-like restoring force. The particle is initially at rest, so $x_0 = x(0) = 0$ and $v_0 = \dot{x}(0) = 0$. At $t = 0$ we suddenly begin to apply a driving force $f(t) = mA\cos(\omega t)$ to the particle, where $A > 0$ is a constant.

For $t > 0$, the particle exhibits some transient motion but after a long time it settles down into a regular steady-state oscillation.

Note for (a): transient part means that we try to get the "homogeneous" linear differential equations, i.e. by assuming that the driving force is zero (the drag force still exists)

(a) Write down $F = ma$ for this system and show that the *transient part* $x_t(t)$ of the total displacement $x(t)$ of the particle is given by

$$x_t(t) = C + \left(\frac{D}{\beta}\right)e^{-\beta t} \quad , \qquad (1)$$

where $\beta = b/m$ and C and D are constants to be determined later from the initial conditions.

(b) We learned in class that the long term, or steady state, part of the motion is found from the *particular solution* $x_p(t)$ of the forced equation of motion.

Find a particular solution by "complexifying" the equation of motion by assuming a complex applied force $\tilde{f}(t) = mAe^{i\omega t}$, so that $f = \Re[\tilde{f}]$, where $\Re[\cdot]$ means "take the real part". Using a clever guess, find a complex expression $\tilde{x}_p(t)$ for the long-term, steady-state motion.

(c) By taking the real part of the complex expression you found in the previous part, find the real steady state motion $x_p(t) = \Re[\tilde{x}_p(t)]$. Write it in the following form

$$x_p(t) = B\cos(\omega t + \phi) \quad , \qquad (2)$$

finding expressions for the real amplitude B and phase ϕ.

HINT: It will be useful to write $\tilde{x}_p$ in the polar form $Be^{i\phi}e^{i\omega t}$ first. $Be^{i\phi}$ is the complex *phasor* of the motion. Draw a phasor diagram in the complex plane.

You may or may not recall that
$\cos(\alpha + \delta) = \cos(\alpha)\cos(\delta) - \sin(\alpha)\sin(\delta)$.

(d) Finally, combine the real transient and the steady-state motion to get the full motion for all time, $x(t) = x_t(t) + x_p(t)$. Apply the initial conditions and determine the constants C and D. You may assume the result of the previous part and just write them in terms of B and ϕ.

PHY254 PS2

Q4) $F_{net} = F_{drive} + F_{drag}$

$$m\ddot{x} = mA\cos(\omega t) - b\dot{x}$$

$$\Rightarrow \ddot{x} + \frac{b}{m}\dot{x} = A\cos(\omega t)$$

a) Transient motion: no driving force

$$\ddot{x} + \frac{b}{m}\dot{x} = 0. \qquad \text{Let } \beta = \frac{b}{m}.$$

Let $p = \dot{x}$, and guess $p = Ke^{rt}$ for a constant K.

$$\dot{p} + \beta p = 0 \quad \Rightarrow \quad Kr\cancel{e^{rt}} + \beta K\cancel{e^{rt}} = 0$$

$$\Rightarrow \quad r + \beta = 0, \quad \text{thus } r = -\beta.$$

Then $p = \dot{x} = Ke^{-\beta t}$.

$$x = \int \dot{x}\, dt = \int Ke^{-\beta t}\, dt$$

$$= -\frac{K}{\beta}e^{-\beta t} + C.$$ Let $D = -K$, another constant. Then

$$\boxed{x_t(t) = C + \left(\frac{D}{\beta}\right)e^{-\beta t}}$$, as required.

b) Steady state motion: guess a solution

$$\ddot{x} + \beta\dot{x} = A\cos(\omega t) \rightsquigarrow \ddot{\tilde{x}} + \beta\dot{\tilde{x}} = Ae^{i\omega t}$$
complexified solution.

Guess $\tilde{x} = Be^{i(\omega t + \phi)}$, for some initial phase ϕ and constant B.

Let $\alpha = Be^{i\phi}$. Then $\tilde{x}(t) = \alpha e^{i\omega t}$.

$\Rightarrow \quad \dot{\tilde{x}} = \alpha i\omega e^{i\omega t}$

and $\quad \ddot{\tilde{x}} = -\alpha \omega^2 e^{i\omega t}$

Our complex ODE then becomes

$$-\alpha\omega^2 \cancel{e^{i\omega t}} + i\beta\omega\alpha \cancel{e^{i\omega t}} = A\cancel{e^{i\omega t}}$$

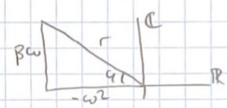
$$\Rightarrow \quad -\alpha\omega^2 + i\beta\omega\alpha = A.$$

Now we can solve for α:

$$\alpha(i\beta\omega - \omega^2) = A \quad \Rightarrow \quad \alpha = \frac{A}{i\beta\omega - \omega^2} = Be^{i\phi}.$$

Notice that $i\beta\omega - \omega^2$ is complex, hence it can be written in complex polar form $re^{i\varphi}$ for some r and φ.

Notice



$$r = \sqrt{A^2 + B^2} = \sqrt{\beta^2\omega^2 + \omega^4}$$

Thus $\quad \varphi = \arctan\left(-\frac{\beta\omega}{\omega^2}\right) = -\arctan\left(\frac{\beta}{\omega}\right).$

This implies

$$\alpha = \frac{A}{r} e^{-i\phi} = \frac{A}{\omega\sqrt{\beta^2+\omega^2}} e^{\arctan\left(\frac{\beta}{\omega}\right)} = Be^{i\phi}$$

Therefore $\quad B = \frac{A}{\omega\sqrt{\beta^2+\omega^2}} \quad$ and $\quad \phi = \arctan\left(\beta/\omega\right).$

Therefore our solution is

$$\boxed{\begin{aligned} &\tilde{x}(t) = B\, e^{i(\omega t + \phi)} \\ &\text{where } B = \frac{A}{\omega\sqrt{\beta^2+\omega^2}} \quad \text{and} \quad \phi = \arctan\left(\frac{\beta}{\omega}\right) \end{aligned}}$$

missing a minus sign

c) Our previous solution was

$$\tilde{x}_p(t) = Be^{i(\omega t + \varphi)}$$

where $B = \frac{A}{\omega\sqrt{\beta^2+\omega^2}}$ and $\varphi = \arctan\left(\frac{\beta}{\omega}\right)$.

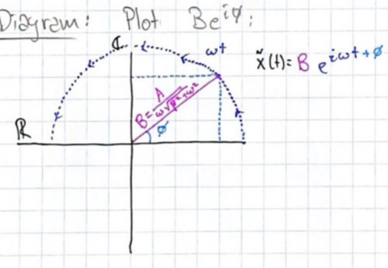
Expanding, we have

$$\tilde{x}_p(t) = B\left[\cos(\omega t + \varphi) + i\sin(\omega t + \varphi)\right]$$

So $x_p(t) = \mathrm{Re}\{\tilde{x}_p(t)\} = B\cos(\omega t + \varphi)$.

$$\boxed{x_p(t) = B\cos(\omega t + \varphi) \quad \text{where } B = \frac{A}{\omega\sqrt{\beta^2+\omega^2}} \text{ and } \varphi = \arctan\left(\frac{\beta}{\omega}\right).}$$

<u>Phasor Diagram:</u> Plot $Be^{i\varphi}$:



d) Constructing the solution:

$$x = x_p + x_t$$
$$= B\cos(\omega t + \varphi) + C + \frac{D}{\beta} e^{-\beta t}.$$

$x_0 = 0, \quad \dot{x}_0 = 0.$

$x \quad 0 = B\cos(\cancel{\omega t} + \varphi) + C + \frac{D}{\beta}\cancel{e^{-\beta t}}$

$$= B\cos\varphi + C + \frac{D}{\beta}$$

$\dot{x} \quad 0 = -B\omega\sin(\cancel{\omega t} + \varphi) - D\cancel{e^{-\beta t}}$

$$= -B\omega\sin\varphi - D$$
$$\Rightarrow D = -B\omega\sin\varphi$$
$$\Rightarrow C = -B\cos\varphi + \frac{B\omega\sin\varphi}{\beta}$$

Thus our solution is

$$\boxed{x(t) = B\cos(\omega t + \varphi) - B\cos\varphi + \frac{B\omega\sin\varphi}{\beta} - \frac{B\omega\sin\varphi}{\beta}e^{-\beta t}}$$

where $\beta = \frac{b}{m}$, $B = \frac{A}{\omega\sqrt{\beta^2+\omega^2}}$, and $\varphi = \arctan\left(\frac{\beta}{\omega}\right)$