PHY 256 PS 5

a) i) $|0\rangle$.

$$\langle\hat{x}\rangle = \langle 0|\hat{x}|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle 0|(a^\dagger + a)|0\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\langle 0|\left[a^\dagger|0\rangle + \overset{0}{a|0\rangle}\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\langle 0|\sqrt{1}|1\rangle = \boxed{0}$$

$$\langle\hat{x}^2\rangle = \langle 0|\hat{x}^2|0\rangle = \frac{\hbar}{2m\omega}\langle 0|(a^\dagger+a)(a^\dagger+a)|0\rangle$$

$$= \frac{\hbar}{2m\omega}\langle 0|\left[(a^\dagger+a)(\sqrt{1}|1\rangle + 0)\right]$$

$$= \frac{\hbar}{2m\omega}\left(\langle 0|a^\dagger|1\rangle + \langle 0|a|1\rangle\right)$$

$$= \frac{\hbar}{2m\omega}\left(\sqrt{2}\langle 0|2\rangle + \sqrt{1}\langle 0|0\rangle\right)$$

$$\boxed{= \frac{\hbar}{2m\omega}}$$

Then $\boxed{\Delta x = \sqrt{\frac{\hbar}{2m\omega}}}$

$$\langle\hat{p}\rangle = \langle 0|\hat{p}|0\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\langle 0|(a^\dagger - a)|0\rangle$$

$$= i\sqrt{\frac{\hbar m\omega}{2}}\langle 0|(\sqrt{1}|1\rangle - 0)\ \boxed{=0}$$

$$\langle\hat{p}^2\rangle = \langle 0|\hat{p}^2|0\rangle = -\frac{\hbar m\omega}{2}\langle 0|\left[(a^\dagger - a)(\sqrt{1}|1\rangle - 0)\right]$$

$$= -\frac{\hbar m\omega}{2}\langle 0|\left[\sqrt{2}|2\rangle - \sqrt{1}|0\rangle\right]$$

$$= (-1)-\frac{\hbar m\omega}{2} = \boxed{\frac{\hbar m\omega}{2}}$$

Then $\boxed{\Delta P = \sqrt{\frac{\hbar m\omega}{2}}}$.

The Heisenberg uncertainty principle is satisfied, since

$$\boxed{\Delta x\,\Delta p = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{\hbar m\omega}{2}} = \frac{\hbar}{2}}.$$

ii) $|1\rangle$.

$$\langle \hat{x} \rangle = \langle 1|\hat{x}|1\rangle = \sqrt{\frac{\hbar}{2m\omega}}\ \langle 1|(a^{+}+a)|1\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\ \langle 1|\left[\sqrt{2}|2\rangle + \sqrt{1}|0\rangle\right]$$

$$\boxed{=0}$$

$$\langle \hat{x}^2 \rangle = \langle 1|\hat{x}^2|1\rangle = \frac{\hbar}{2m\omega}\ \langle 1|(a^{+}+a)(a^{+}+a)|1\rangle$$

$$= \frac{\hbar}{2m\omega}\ \langle 1|(a^{+}+a)\left[\sqrt{2}|2\rangle + \sqrt{1}|0\rangle\right]$$

$$= \frac{\hbar}{2m\omega}\ \langle 1|\left[\sqrt{2\cdot 3}|3\rangle + \sqrt{2\cdot 2}|1\rangle + \sqrt{1}|1\rangle + \cancel{a|0\rangle}^{0}\right]$$

$$= \frac{\hbar}{2m\omega}\ [2+1] = \boxed{\frac{3\hbar}{2m\omega}}.$$

Then $\boxed{\Delta x = \sqrt{\frac{3\hbar}{2m\omega}}}$.

$$\langle \hat{p} \rangle = \langle 1|\hat{p}|1\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\ \langle 1|(a^{+}-a)|1\rangle$$

$$= i\sqrt{\frac{\hbar m\omega}{2}}\ \langle 1|\left[\sqrt{2}|2\rangle - \sqrt{1}|0\rangle\right] \quad \boxed{=0}$$

$$\langle \hat{p}^2 \rangle = \langle 1|\hat{p}^2|1\rangle = -\frac{\hbar m\omega}{2}\ \langle 1|(a^{+}-a)(a^{+}-a)|1\rangle$$

$$= -\frac{\hbar m\omega}{2}\ \langle 1|(a^{+}-a)\left[\sqrt{2}|2\rangle - \sqrt{1}|0\rangle\right]$$

$$= -\frac{\hbar m\omega}{2}\ \langle 1|\left[\sqrt{2\cdot 3}|3\rangle - \sqrt{1}|1\rangle - \sqrt{2\cdot 2}|1\rangle + \cancel{a|0\rangle}^{0}\right]$$

$$= -\frac{\hbar m\omega}{2}\ (-3)$$

$$\boxed{= \frac{3\hbar m\omega}{2}}$$

Then $\boxed{\Delta P = \sqrt{\frac{3\hbar m\omega}{2}}}$.

This satisfies the uncertainty principle, since

$$\boxed{\Delta x \Delta p = \sqrt{\frac{3\hbar m\omega}{2}}\sqrt{\frac{3\hbar}{2m\omega}} = \frac{3\hbar}{2} > \frac{\hbar}{2}}.$$

D) i) $\frac{1}{\sqrt{2}}(|8\rangle + e^{i\phi}|9\rangle)$.

$$\langle \hat{x} \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\frac{\langle 8| + e^{-i\phi}\langle 9|}{\sqrt{2}} \right) (a^\dagger + a) \left(\frac{|8\rangle + e^{i\phi}|9\rangle}{\sqrt{2}} \right)$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}} \left(\langle 8| + e^{-i\phi}\langle 9| \right) \left[\sqrt{9}|9\rangle + e^{i\phi}\sqrt{10}|10\rangle + \sqrt{8}|7\rangle + e^{i\phi}\sqrt{9}|8\rangle \right]$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}} \left(\langle 8| + e^{-i\phi}\langle 9| \right) \left(3|9\rangle + e^{i\phi}\cdot 3|8\rangle \right)$$

(get rid of orthogonal states for simplicity)

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}} \left[3e^{i\phi} + 3e^{-i\phi} \right]$$

$$\boxed{= 3\sqrt{\frac{\hbar}{2m\omega}} \cos\phi}$$

$$\langle \hat{x}^2 \rangle = \frac{\hbar}{2m\omega} \left(\frac{\langle 8| + e^{-i\phi}\langle 9|}{\sqrt{2}} \right) (a^\dagger + a)(a^\dagger + a) \left(\frac{|8\rangle + e^{i\phi}|9\rangle}{\sqrt{2}} \right)$$

$$= \frac{\hbar}{4m\omega} \left(\langle 8| + e^{-i\phi}\langle 9| \right) (a^\dagger + a) \left[3|9\rangle + e^{i\phi}\sqrt{10}|10\rangle + \sqrt{8}|7\rangle + e^{i\phi} 3|8\rangle \right]$$

$$= \frac{\hbar}{4m\omega} \left(\langle 8| + e^{-i\phi}\langle 9| \right) \left[3\sqrt{10}|10\rangle + e^{i\phi}\sqrt{10\cdot 11}|11\rangle + \sqrt{8\cdot 8}|8\rangle + e^{i\phi} 3\cdot\sqrt{9}|9\rangle \right.$$
$$\left. + 3\cdot\sqrt{9}|8\rangle + e^{i\phi}\sqrt{10\cdot 10}|9\rangle + \sqrt{8\cdot 7}|6\rangle + e^{i\phi} 3\sqrt{8}|7\rangle \right]$$

$$= \frac{\hbar}{4m\omega} \left(\langle 8| + e^{-i\phi}\langle 9| \right) \left(8|8\rangle + e^{i\phi} 9|9\rangle + 9|8\rangle + e^{i\phi} 10|9\rangle \right)$$

(get rid of orthogonal parts for simplicity)

$$= \frac{\hbar}{4m\omega} (8 + 9 + 9 + 10)$$

$$= 9\frac{\hbar}{m\omega}$$

Thus $\Delta x = \sqrt{\frac{9\hbar}{m\omega} - \frac{9\hbar}{2m\omega}\cos^2\phi} = 3\sqrt{\frac{\hbar}{2m\omega}}\sqrt{1-\cos^2\phi}$

$$\boxed{\Delta x = 3\sqrt{\frac{\hbar}{2m\omega}} \sin\phi}$$

ii) $\frac{1}{\sqrt{2}}\left(|1\rangle + e^{i\phi}|3\rangle\right)$.

$$\langle \hat{x}\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(\langle 1| + e^{-i\phi}\langle 3|\right)\left[a^{\dagger} + a\right]\left(|1\rangle + e^{i\phi}|3\rangle\right)$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(\langle 1| + e^{-i\phi}\langle 3|\right)\left[\sqrt{2}|2\rangle + e^{i\phi}\sqrt{4}|4\rangle + \sqrt{1}|0\rangle + e^{i\phi}\sqrt{3}|2\rangle\right]$$

$\boxed{= 0}$ by orthogonality.

$$\langle \hat{x}^2\rangle = \frac{\hbar}{4m\omega}\left(\langle 1| + e^{-i\phi}\langle 3|\right)\left[a^{\dagger}+a\right]\left[a^{\dagger}+a\right]\left(|1\rangle + e^{i\phi}|3\rangle\right)$$

$$= \frac{\hbar}{4m\omega}\left(\langle 1| + e^{-i\phi}\langle 3|\right)\left[a^{\dagger}+a\right]\left(\sqrt{2}|2\rangle + e^{i\phi}\,2|4\rangle + |0\rangle + e^{i\phi}\sqrt{3}|2\rangle\right)$$

$$= \frac{\hbar}{4m\omega}\left(\langle 1| + e^{-i\phi}\langle 3|\right)\Big[\sqrt{2\cdot 3}|3\rangle + e^{i\phi}\,2\sqrt{5}|5\rangle + |1\rangle + e^{i\phi}\sqrt{3\cdot 3}|3\rangle$$

$$+ \sqrt{2\cdot 2}|1\rangle + e^{i\phi}\sqrt{4}\cdot 2|3\rangle + 0 + e^{i\phi}\sqrt{3\cdot 2}|1\rangle\Big]$$

$$= \frac{\hbar}{4m\omega}\left(\langle 1| + e^{-i\phi}\langle 3|\right)\left(\sqrt{6}|3\rangle + |1\rangle + 3e^{i\phi}|3\rangle + 4|1\rangle + 4e^{i\phi}|3\rangle + e^{i\phi}\sqrt{6}|1\rangle\right]$$

$$= \frac{\hbar}{4m\omega}\left(\langle 1| + e^{-i\phi}\langle 3|\right)\left((5+\sqrt{6}e^{i\phi})|1\rangle + (\sqrt{6}+7e^{i\phi})|3\rangle\right)$$

$$= \frac{\hbar}{4m\omega}\left(5+\sqrt{6}e^{i\phi} + \sqrt{6}e^{-i\phi} + 7\right)$$

$$= \frac{3\hbar}{m\omega}\sqrt{6}\left(e^{i\phi} + e^{-i\phi}\right)$$

$$= \frac{3\hbar}{m\omega}2\cdot\sqrt{6}\cos\phi$$

$$= \frac{6\sqrt{6}\,\hbar}{m\omega}\cos\phi$$

Then $\boxed{\Delta x = \sqrt{\frac{6\sqrt{6}\,\hbar}{m\omega}\cos\phi}}$

PHY256 PS5

Q2) a) Arbitrary state: $\boxed{|\psi\rangle = a|0\rangle + be^{i\phi}|1\rangle}$

b) For normalization, we require that $|a|^2 + |b|^2 = 1$.

This implies that $a, b \in \mathbb{R}$.

Since $|e^{i\phi}| = 1$, we require two real numbers:

$\sin\phi$, $\cos\phi$ to describe $\cos\phi + i\sin\phi$.

Thus, we require 4 real numbers.

c) $\boxed{|\psi\rangle = (a|0\rangle + be^{i\phi_1}|1\rangle) \otimes (c|0\rangle + de^{i\phi_2}|1\rangle)}$

$= ac|00\rangle + ade^{i\phi_2}|01\rangle + bce^{i\phi_1}|10\rangle + bde^{i(\phi_1+\phi_2)}|11\rangle$.

d) We require 4 numbers for normalization: $a,b,c,d \in \mathbb{R}$.

Likewise, 4 real numbers to describe $e^{i\phi_1}$, $e^{i\phi_2}$, $\Rightarrow$ $\boxed{8\ \mathbb{R}\text{ numbers}}$.

e) $\boxed{|\psi\rangle = a|00\rangle + be^{i\phi_1}|10\rangle + ce^{i\phi_2}|01\rangle + de^{i\phi_3}|11\rangle}$

f) $\boxed{10\text{ real numbers}}$ Equivalent reasoning to (d).

g) Seperable state of 10 qubits: $\boxed{2^{10} + 2\cdot 10 \text{ real numbers}}$

Fully general: $2^{10} + 2\cdot(2^{10}-1) = \boxed{2^{12} - 2 \text{ real numbers}}$

For N qubit seperable state: $\underbrace{2^N}_{\text{normalization}} + 2\cdot N.]$ phasors.

Non-seperable / fully general: $\underbrace{2^N}_{\text{Normalization}} + 2\cdot(2^N-1)]$ phasors.

20 qubits:

$$\boxed{\begin{array}{l} \text{Seperable: } 2^{20} + 40 \text{ real numbers.} \\ \text{General: } 2^{20} + 2\cdot(2^{20}-1) \text{ real numbers} \end{array}}$$

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Q3) In an infinite square well, we have

$$\varphi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, \dots$$

where $L = a$ in this case.

We have that $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) + \frac{1}{\sqrt{2}}\sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right)$.

For time evolution, $e^{-iE_n t/\hbar}$, $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$, $n = 1, 2$.

Then $$|\psi(t)\rangle = \frac{1}{\sqrt{a}} e^{-i\frac{\pi^2\hbar t}{2ma^2}} \sin\left(\frac{\pi x}{a}\right) + \frac{1}{\sqrt{a}} e^{-i\frac{2\pi^2\hbar t}{ma^2}} \sin\left(\frac{2\pi x}{a}\right)$$

in position basis.

Find $\langle \hat{p} \rangle$ as a function of time. $\hat{p} = -i\hbar \frac{\partial}{\partial x}$.

Then $$\langle \hat{p} \rangle = -i\hbar \int_0^a dx \left[\frac{1}{\sqrt{a}} e^{i\frac{\pi^2\hbar t}{2ma^2}} \sin\left(\frac{\pi x}{a}\right) + \frac{1}{\sqrt{a}} e^{i\frac{2\pi^2\hbar t}{ma^2}} \sin\left(\frac{2\pi x}{a}\right)\right]$$
$$\cdot \frac{\partial}{\partial x}\left[\frac{1}{\sqrt{a}} e^{-i\frac{\pi^2\hbar t}{2ma^2}} \sin\left(\frac{\pi x}{a}\right) + \frac{1}{\sqrt{a}} e^{-i\frac{2\pi^2\hbar t}{ma^2}} \sin\left(\frac{2\pi x}{a}\right)\right]$$

$$= -\frac{i\hbar}{a} \int_0^a dx \left[e^{i\frac{\pi^2\hbar t}{2ma^2}} \sin\left(\frac{\pi x}{a}\right) + e^{i\frac{2\pi^2\hbar t}{ma^2}} \sin\left(\frac{2\pi x}{a}\right)\right]$$
$$\cdot \left[\frac{\pi}{a} e^{-i\frac{\pi^2\hbar t}{2ma^2}} \cos\left(\frac{\pi x}{a}\right) + \frac{2\pi}{a} e^{-i\frac{2\pi^2\hbar t}{ma^2}} \cos\left(\frac{2\pi x}{a}\right)\right]$$

$$= -\frac{i\hbar}{a} \int_0^a dx \Big[\frac{\pi}{a} \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right) + \frac{2\pi}{a} \sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{2\pi x}{a}\right)$$
$$+ \frac{\pi}{a} e^{i\left(\frac{2\pi^2\hbar t}{ma^2} - \frac{\pi^2\hbar t}{2ma^2}\right)} \sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right) + \frac{2\pi}{a} e^{i\left(\frac{\pi^2\hbar t}{2ma^2} - \frac{2\pi^2\hbar t}{ma^2}\right)}$$
$$\cdot \cos\left(\frac{2\pi x}{a}\right) \sin\left(\frac{\pi x}{a}\right)\Big]$$

By the orthogonality of sin, cos, we have

$$\langle \hat{p} \rangle = -\frac{i\hbar}{a^2}\left[\int_0^a dx\ \pi e^{i\frac{3\pi^2\hbar t}{2ma^2}} \sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right) + 2\pi e^{-i\frac{3\pi^2\hbar t}{2ma^2}} \cos\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\right]$$
$$= -\frac{i\pi\hbar}{a^2} e^{i\frac{3\pi^2\hbar t}{2ma^2}} \int_0^a dx\ \sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right) - \frac{i2\pi\hbar}{a^2} e^{-i\frac{3\pi^2\hbar t}{2ma^2}} \int_0^a dx \cos\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right).$$

Aside, our trig identities are

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right].$$

Then $\displaystyle\int_0^a dx\, \sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right) = \frac{1}{2}\int_0^a dx\, \sin\left(\frac{3\pi x}{a}\right) + \sin\left(\frac{\pi x}{a}\right)$

$$= -\frac{1}{2}\left[\frac{a}{3\pi}\cos\left(\frac{3\pi x}{a}\right)\Big|_0^a + \frac{a}{\pi}\cos\left(\frac{\pi x}{a}\right)\Big|_0^a\right]$$

$$= -\frac{1}{2}\left[\frac{-2a}{3\pi} + \frac{-2a}{\pi}\right] = \frac{4a}{3\pi}.$$

Similarly, $\displaystyle\int_0^a dx\, \cos\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right) = \frac{1}{2}\left(\frac{2a}{3\pi} - \frac{2a}{\pi}\right) = -\frac{2a}{3\pi}.$

Then

$$\langle\hat{p}\rangle = -i\hbar\left[\frac{\pi}{a^2}\cdot\frac{4a}{3\pi}\, e^{i\frac{3\pi^2\hbar t}{2ma^2}} - \frac{2\pi}{a^2}\cdot\frac{2a}{3\pi}\, e^{-i\frac{3\pi^2\hbar t}{2ma^2}}\right].$$

Let $\omega = \dfrac{3\pi^2\hbar}{2ma^2}$.

Then $\displaystyle\langle\hat{p}\rangle = -i\hbar\left[\frac{4}{3a}e^{i\omega t} - \frac{4}{3a}e^{-i\omega t}\right]$

$$= \cancel{i}\hbar\frac{4}{3a}\left(\cancel{\cos\omega t} + \cancel{i}\sin\omega t - \cancel{\cos\omega t} \cancel{+} i\sin\omega t\right)$$

$$= \frac{8}{3a}\sin\omega t.$$

Therefore $\boxed{\langle\hat{p}\rangle = \frac{8}{3a}\sin\left(\frac{3\pi^2\hbar}{2ma^2}t\right)}$

PHY256 PS5

Q4) a) In the position basis, we have an integral

$$\boxed{\langle m|\hat{x}^2|n\rangle \rightarrow \int_{-\infty}^{\infty} dx\, \psi_m^*(x)\, \hat{x}^2\, \psi_n(x). \neq 0.}$$

b) We have that $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(a^\dagger + a)$.

So

$$\langle m|\hat{x}^2|n\rangle = \frac{\hbar}{2m\omega}\, \langle m|(a^\dagger+a)(a^\dagger+a)|n\rangle$$

$$= \frac{\hbar}{2m\omega}\, \langle m|(a^\dagger+a)\left[\sqrt{n+1}\,|n+1\rangle + \sqrt{n}\,|n-1\rangle\right]$$

$$= \frac{\hbar}{2m\omega}\, \langle m|\left[\sqrt{n+1}\sqrt{n+2}\,|n+2\rangle + \sqrt{n}\sqrt{n}\,|n\rangle + \sqrt{n+1}\sqrt{n+1}\,|n\rangle + \sqrt{n}\sqrt{n-1}\,|n-2\rangle\right]$$

$$= \frac{\hbar}{2m\omega}\left[\sqrt{(n+1)(n+2)}\,\langle m|n+2\rangle + n\,\langle m|n\rangle + (n+1)\,\langle m|n\rangle + \sqrt{n(n-1)}\,\langle m|n-2\rangle\right]$$

For $\langle m|\hat{x}^2|n\rangle \neq 0$, we require that either

$$\boxed{\begin{array}{l} |m\rangle = |n-2\rangle \\ |m\rangle = |n\rangle \quad \text{or} \\ |m\rangle = |n+2\rangle. \end{array}}$$

PHY256 · PS5

Q5) We know that a general state $|\psi\rangle$ can be expended in terms of the energy basis:

$$|\psi\rangle = \sum_n C_n |E_n\rangle .$$

With time dependence,

$$|\psi(t)\rangle = \sum_n C_n e^{-i\frac{E_n t}{\hbar}} |E_n\rangle .$$

Our energies are given:

$$\mathcal{H}|S\rangle = E_S|S\rangle \quad \text{and} \quad \mathcal{H}|A\rangle = E_A|A\rangle .$$

Thus

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} C_1 e^{-i\frac{E_S t}{\hbar}} \underbrace{[|L\rangle + |R\rangle]}_{|S\rangle} + \frac{1}{\sqrt{2}} C_2 e^{-i\frac{E_A t}{\hbar}} \underbrace{[|L\rangle - |R\rangle]}_{|A\rangle}$$

For sake of simplicity, let $\omega_S = \frac{E_S}{\hbar}$, $\omega_A = \frac{E_A}{\hbar}$.

Then $$|\psi(t)\rangle = \frac{C_1}{\sqrt{2}}(\cos\omega_S t - i\sin\omega_S t)|L\rangle + \frac{C_1}{\sqrt{2}}(\cos\omega_S t - i\sin\omega_S t)|R\rangle + \frac{C_2}{\sqrt{2}}(\cos\omega_A t - i\sin\omega_A t)|L\rangle - \frac{C_2}{\sqrt{2}}(\cos\omega_A t - i\sin\omega_A t)|R\rangle$$

$$= \frac{1}{\sqrt{2}}\left[C_1 e^{-i\omega_S t} + C_2 e^{-i\omega_A t}\right]|L\rangle + \frac{1}{\sqrt{2}}\left[C_1 e^{-i\omega_S t} - C_2 e^{-i\omega_A t}\right]|R\rangle .$$

At $t=0$, $|\psi(0)\rangle = |L\rangle$.

This implies that $C_1 = C_2$ and $\frac{C_1+C_2}{\sqrt{2}} = \frac{2C_1}{\sqrt{2}} = 1$, thus $C_1 = C_2 = \frac{1}{\sqrt{2}}$.

Thus

$$\boxed{|\psi(t)\rangle = \frac{1}{2}\left(e^{-i\frac{E_S}{\hbar}t} + e^{-i\frac{E_A}{\hbar}t}\right)|L\rangle + \frac{1}{2}\left(e^{-i\frac{E_S}{\hbar}t} - e^{-i\frac{E_A}{\hbar}t}\right)|R\rangle}$$

We have that $\langle L|\Psi(t)\rangle = \frac{1}{2}\left(e^{-i E_S t/\hbar} + e^{-i E_A t/\hbar}\right)$..

Thus $|\langle L|\Psi(t)\rangle|^2 = \frac{1}{4}\left|e^{-i \frac{E_S}{\hbar}t} + e^{-i \frac{E_A}{\hbar}t}\right|^2$.

The probability of measuring $|L\rangle$ is going to time-dependent.

Then

$$|\langle L|\Psi(t)\rangle|^2 = \frac{1}{4}\left(e^{i\omega_S t} + e^{i\omega_A t}\right)\left(e^{-i\omega_S t} + e^{-i\omega_A t}\right)$$

$$= \frac{1}{4}\left(1+1 + e^{i(\omega_S-\omega_A)t} + e^{-i(\omega_S-\omega_A)t}\right)$$

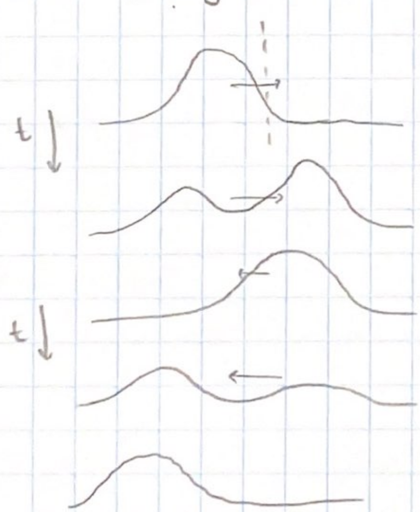
$$= \frac{1}{4}\left(2 + 2\cos\left((\omega_S-\omega_A)t\right)\right)$$

$$\boxed{\mathbb{P}_{|L\rangle} = \frac{1}{4}\left(2 + 2\cos\left(\frac{E_S-E_A}{\hbar}t\right)\right)}$$

Meaning, physically.

- At $t=0$, $\mathbb{P}_{|L\rangle} = 100\%$, as it should be.
- Whenever $\frac{E_S-E_A}{\hbar}t = K\cdot 2\pi$, 100% probability of measuring $|L\rangle$.
- Whenever $\frac{E_S-E_A}{\hbar}t = K\cdot 2\pi + \pi$, 0% probability of measuring $|L\rangle$.
- Whenever $\frac{E_S-E_A}{\hbar}t = K\cdot \pi + \frac{\pi}{2}$, 50% probability of measuring $|L\rangle$.

$\Longrightarrow$ oscillatory motion in the wells. May look like



PHY256 PS5

Q6) a) $x+y$: Seperable. Let $V_x = x$, $V_y = y$.

Then $V = x+y = V_x + V_y$, in cartesian coordinates,

since x and y are independent of each other.

b) $e^{-(x^2+y^2)/2a^2}$: Seperable in polar coordinates. Let $r^2 = x^2+y^2$.

Then, this reduces the two dimensional potential into a potential in polar coordinates only dependent on r, not θ.

We have $V(r) = e^{-r^2/2a^2}$.

c) $0 \iff |x| < a$, $|y| < a$. Not seperable. In either cartesian, polar.

A two-dimensional finite square well with $-a < x < a$ and $-a < y < a$. In this case, x's potential depends on y and vice versa. For instance, $x \in (-a, a)$ but $y > a$, which would result in the potential being V_0.

Since this a two-dimensional well, it is neither seperable in polar coordinates. The square in polar coordinates would both depend on r and θ.

d) $x^2+y^2 < a^2$. Seperable in polar and cartesian coordinates.

In polar, let $r^2 = x^2+y^2$. Then $r^2 < a^2$, not dependent on θ.

Similarly, both $x^2 < \frac{1}{2}a^2$ and $y^2 < \frac{1}{2}a^2$, so $x^2+y^2 < a^2$, which is seperable in cartesian coordinates.

e) $\sqrt{x^2+y^2}$. Seperable in polar coordinates

Let $r^2 = x^2+y^2$, but then $r = \pm\sqrt{x^2+y^2}$.

Restricting r to be positive, our potential becomes

equivalent to $U(x)$ as described in cartesian coordinates,

$r = \sqrt{x^2+y^2}$.

This is not seperable in cartesian coordinates, mainly because

of high school math: $\sqrt{x^2+y^2} \neq x+y$.

PHY256 PS5

Q7) a) Since $\mathcal{H} = H_x + H_y$, $E = E_x + E_y$.

Then $E = (n_x + \frac{1}{2})\hbar\omega + (n_y + \frac{1}{2})\hbar\omega$

where n_x, n_y are the energies of x, y, respectively.

$|0\rangle_x |0\rangle_y \Rightarrow n_x = n_y = 0$, so $\boxed{E_{00} = \hbar\omega}$

$|0\rangle_x |1\rangle_y \Rightarrow n_x = 0,\ n_y = 1$, so $\boxed{E_{01} = 2\hbar\omega}$

$|1\rangle_x |1\rangle_y \Rightarrow n_x = n_y = 1$, so $\boxed{E_{11} = 3\hbar\omega.}$

b) The energies of $|10\rangle$ and $|01\rangle$ are the same, being $E_{10} = E_{01} = 2\hbar\omega$.

By Schrödinger time dependence, the observables evolve at $e^{-iEt/\hbar}$, so we have that the observables evolve with $e^{-i2\omega t}$, therefore

observables evolve with a frequency of 2ω.

c) Given
$$a_{\pm 45} = \frac{a_x \pm a_y}{\sqrt{2}},$$ we can rearrange:

$$a_x = \frac{a_{+45} + a_{-45}}{\sqrt{2}} \quad \text{and} \quad a_y = \frac{a_{+45} - a_{-45}}{\sqrt{2}}.$$

The Hermitian conjugate is linear, so

$$a_x^\dagger = \frac{a_{+45}^\dagger + a_{-45}^\dagger}{\sqrt{2}} \quad \text{and} \quad a_y^\dagger = \frac{a_{+45}^\dagger - a_{-45}^\dagger}{\sqrt{2}}$$

Since the Hamiltonian is given by

$$\mathcal{H} = H_x + H_y$$

$$= \hbar\omega\left(a_x^\dagger a_x + \tfrac{1}{2}\right) + \hbar\omega\left(a_y^\dagger a_y + \tfrac{1}{2}\right).$$

Substituting, we have

$$\boxed{\mathcal{H} = \hbar\omega\left[\frac{a_{+45}^\dagger + a_{-45}^\dagger}{\sqrt{2}} \cdot \frac{a_{+45} + a_{-45}}{\sqrt{2}} + \frac{a_{+45}^\dagger - a_{-45}^\dagger}{\sqrt{2}} \cdot \frac{a_{+45} - a_{-45}}{\sqrt{2}} + 1\right]}$$

d) Given $|\psi\rangle = \dfrac{|01\rangle - |10\rangle}{\sqrt{2}}$,

$$a_{+45}|\psi\rangle = \tfrac{1}{2}\left[a_x + a_y\right]\left[|01\rangle - |10\rangle\right]$$

$$= \tfrac{1}{2}\left[\cancel{a_x|01\rangle}^{0} - a_x|10\rangle + a_y|01\rangle - \cancel{a_y|10\rangle}^{0}\right]$$

$$= \tfrac{1}{2}\left[-\sqrt{1}\,|00\rangle + \sqrt{1}\,|00\rangle\right]$$

$$\boxed{= 0}$$

$$a_{-45}|\psi\rangle = \tfrac{1}{2}\left[a_x - a_y\right]\left[|01\rangle - |10\rangle\right]$$

$$= \tfrac{1}{2}\left[\cancel{a_x|01\rangle}^{0} - a_x|10\rangle - a_y|01\rangle + \cancel{a_y|10\rangle}^{0}\right]$$

$$= \tfrac{1}{2}\left[-2\,|00\rangle\right]$$

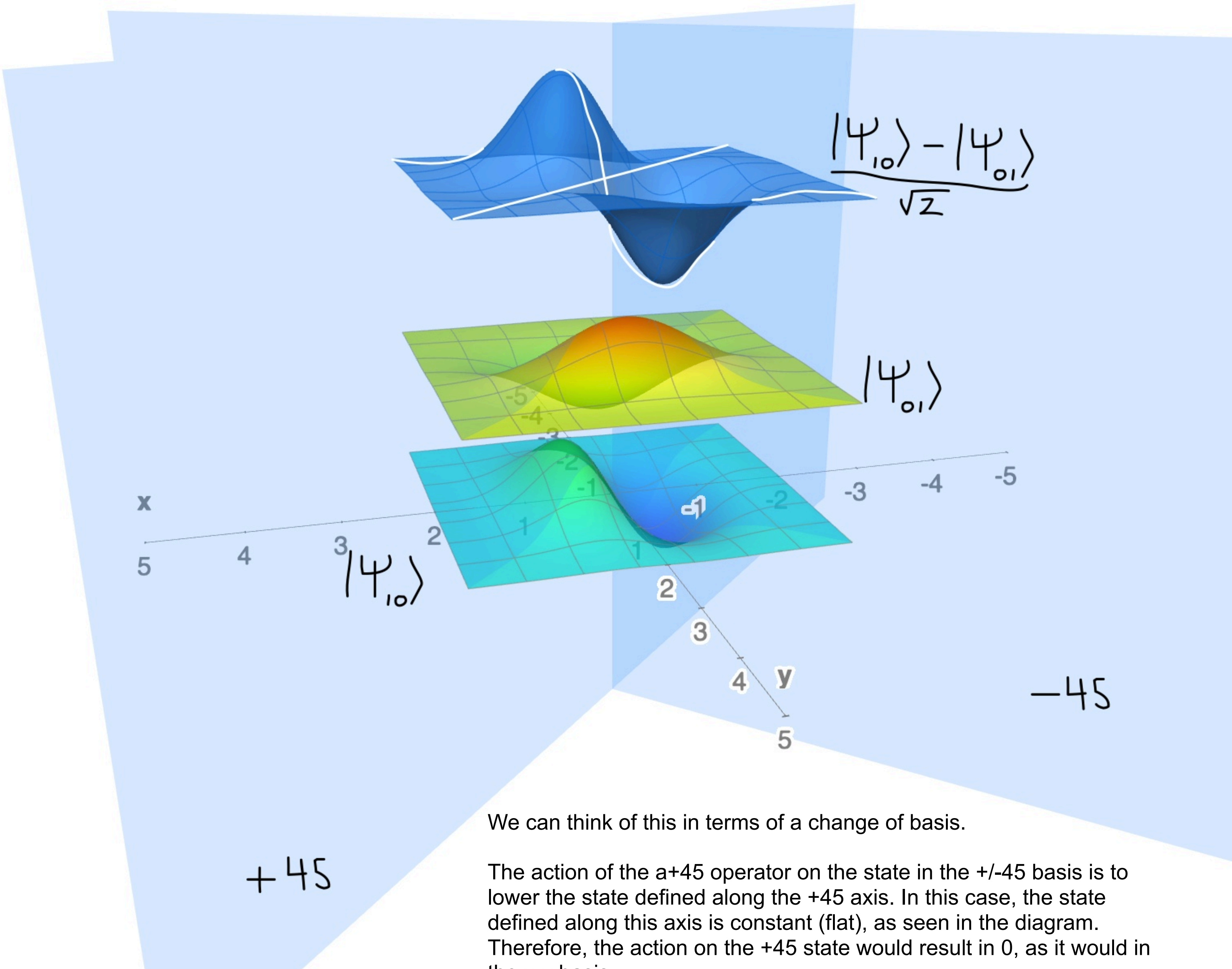
$$\boxed{= -|00\rangle}$$

The physical meaning of these operators are difficult to describe without a visual (so I included one). Essentially, (informally)

a_{+45} lowers x in first
lowers y in second (lowers state along +45° axis)

Please see diagram.

a_{-45} lowers y in first
lowers x in second. (lowers state along −45° axis)



We can think of this in terms of a change of basis.

The action of the a+45 operator on the state in the +/-45 basis is to lower the state defined along the +45 axis. In this case, the state defined along this axis is constant (flat), as seen in the diagram. Therefore, the action on the +45 state would result in 0, as it would in the x-y basis.

Similarly, the action of the a-45 operator on the state in this basis is to lower the state defined along the -45 axis. We can see that the state defined along this axis is the first excited state in this basis, which implies that a-45 operator acts on the wavefunction to lower it to the ground state, |00>, which is does.

In the x-y basis, this is equivalent to having a first excited state (for instance, |10> in the x basis) and applying the a-x lowering operator on it, which would again result in the ground state |00>.