PHY354 PS5 — 04/06/2022

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1. *The fast top:* Consider a heavy symmetric top of mass M, pinned at point P which is a distance ℓ from the centre of mass, as discussed in class. The principal moments of inertia about P are I_1, I_1 and I_3. The top is spun with initial conditions $\dot{\phi} = 0$ and $\theta = \theta_0$. Show that θ obeys the equation of motion

$$I_1\ddot{\theta} = -\frac{dV_{\text{eff}}(\theta)}{d\theta}$$

where

$$V_{\text{eff}}(\theta) = \frac{I_3^2\Omega_3^2}{2I_1}\frac{(\cos\theta - \cos\theta_0)^2}{\sin^2\theta} + Mg\ell\cos\theta.$$

Suppose the top is spinning very fast, so that $I_3\Omega_3 \gg \sqrt{Mg\ell I_1}$. Show that θ_0 is close to the minimum of $V_{\text{eff}}(\theta)$, and use this fact to deduce that the top nutates with frequency $\omega \simeq \Omega_3 I_3/I_1$ and draw the subsequent motion.

I will begin by finding the Lagrangian of the system. Finding the Lagrangian will allow me to determine any conserved quantites and cyclic coordinates, so that I may write the Hamiltonian in terms of these generalized momenta and apply Hamilton's equations. As discussed in class and by the Euler Angles, the total kinetic energy is $T = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + \frac{1}{2}(\dot{\psi} + \dot{\varphi}\cos\theta)^2$. In a gravitational field, and the center of mass a distance ℓ from the pivot point, the potential is given by $V = Mg\ell\cos\theta$. Therefore the Lagrangian of the system is

$$\mathcal{L} = T - V = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + \frac{1}{2}(\dot{\psi} + \dot{\varphi}\cos\theta)^2 - Mg\ell\cos\theta.$$

Notice that the Lagrangian is independent of $\dot{\varphi}$, $\dot{\psi}$ and t, and therefore we have 3 cyclic coordinates hence 3 conserved quantites. They are the generalized momenta $p_\varphi = \dfrac{\partial\mathcal{L}}{\partial\varphi}$ and $p_\psi = \dfrac{\partial\mathcal{L}}{\partial\psi}$:

$$\begin{aligned} p_\varphi &= \dot{\varphi}[I_1\sin^2\theta + I_3\cos^2\theta] + I_3\dot{\psi}\cos\theta \equiv M_z \\ p_\psi &= I_3(\dot{\psi} + \dot{\varphi}\cos\theta) = I_3\Omega_3 \equiv M_3. \end{aligned}$$

These expressions are also derived in the text (and in lecture). With M_z and M_3 being conserved quantities, we are able to expression $\dot{\varphi}$ and $\dot{\psi}$ in terms of these angular momenta. After rearranging, we find that

$$\dot{\varphi} = \frac{M_z - M_3\cos\theta}{I_1\sin^2\theta} \qquad \text{and} \qquad \dot{\psi} = \frac{M_3}{I_3} - \cos\theta\frac{M_z - M - 3\cos\theta}{I_1\sin^2\theta}.$$

Furthermore, the total energy of the system is equivalent to the Hamiltonian, which is simply

$$H = E = T + V = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + \frac{1}{2}(\dot{\psi} + \dot{\varphi}\cos\theta)^2 + Mg\ell\cos\theta.$$

With this, we can express the Lagrangian and the Hamiltonian in terms of the conserved quantities M_z and M_3, since doing this will assist in apply the initial conditions given to the system. After substituting and simplifying, we have that

$$\mathcal{L} = \frac{1}{2}I_1\dot{\theta}^2 + \frac{(M_z - M_3\cos\theta)^2}{2I_1\sin^2\theta} - Mg\ell\cos\theta$$
$$H = \frac{1}{2}I_1\dot{\theta}^2 + \underbrace{\frac{(M_z - M_3\cos\theta)^2}{2I_1\sin^2\theta} + Mg\ell\cos\theta}_{V_{\text{eff}}(\theta)},$$

(all of these expressions are also given in the textbook) which also reduces the problem down to a single dimension of generalized coordinate θ. I will proceed by applying Hamilton's equations for $\dot{p} = -\frac{\partial H}{\partial \theta}$. The generalized momentum p_θ is given by $\frac{\partial \mathcal{L}}{\partial \theta} = I\dot{\theta}$, which then implies that $\dot{p}_\theta = I\ddot{\theta}$. We have that

$$I_1\ddot{\theta} = -\frac{\partial H}{\partial \theta} = 0 - \frac{\partial V_{\text{eff}}(\theta)}{\partial \theta}.$$

Now, allow me to invoke the initial conditions $\dot{\varphi} = 0$ and $\theta_i = \theta_0$. Since M_z and M_3 are conserved, they are constant for all time, so $M_z = I_3\dot{\psi}\cos\theta = I_3\dot{\psi}\cos\theta_0$. With $\dot{\varphi} = 0$, then $\dot{\psi} = \frac{M_3}{I_3}$, which implies that $M_z = M_3\cos\theta_0$. The effective potential can then be written as

$$V_{\text{eff}}(\theta) = \frac{(M_3\cos\theta_0 - M_3\cos\theta)^2}{2I_1\sin^2\theta} + Mg\ell\cos\theta = \frac{M_3^2(\cos\theta_0 - \cos\theta)^2}{2I_1\sin^2\theta} + Mg\ell\cos\theta.$$

Since $M_3 \equiv I_3\Omega_3$ along the $\hat{x}_3$ principle axis, then

$$\boxed{V_{\text{eff}}(\theta) = \frac{I_3^2\Omega_3^2(\cos\theta - \cos\theta_0)^2}{2I_1\sin^2\theta} + Mg\ell\cos\theta \qquad \text{and} \qquad I_1\ddot{\theta} = -\frac{\partial V_{\text{eff}}(\theta)}{\partial\theta}.}$$

For the fast top, we have that $I_3\Omega_3 \gg \sqrt{Mg\ell I_1}$. This implies that $\frac{I_3^2\Omega_3^2}{2I_1Mg\ell} \gg 1$ or $\frac{2I_1Mg\ell}{I_3^2\Omega_3^2} \ll 1$. I will proceed by Taylor expanding to the second order, which gives a good approximation around the minimum of the effective potential, since we are examining small perturbations from a stable orbit. To help me Taylor expand to approximate, I will give the expressions for the first and second derivatives of the effective potential:

$$V'_{\text{eff}}(\theta) = -Mg\ell\sin\theta - \frac{I_3^2\Omega_3^2}{I_1}\frac{\cos\theta - \cos\theta_0}{\sin\theta} - \frac{I_3^2\Omega_3^2}{I_1}\frac{\cos\theta(\cos\theta - \cos\theta_0)^2}{\sin^3\theta}$$
$$\implies V'_{\text{eff}}(\theta_0) = -Mg\ell\sin\theta_0$$

$$V''_{\text{eff}}(\theta) = \frac{I_3^2\Omega_3^2}{I_1} - Mg\ell\cos\theta + \frac{I_3^2\Omega_3^2}{I_1}\frac{(\cos\theta - \cos\theta_0)^2}{\sin^2\theta} + \frac{I_3^2\Omega_3^2}{I_1}\frac{\cos\theta(\cos\theta - \cos\theta_0)}{\sin^2\theta}$$
$$+ \frac{I_3^2\Omega_3^2}{I_1}\frac{\cos^2\theta(\cos\theta - \cos\theta_0)^2}{\sin^4\theta}$$
$$\implies V''_{\text{eff}}(\theta_0) = \frac{I_3^2\Omega_3^2}{I_1} - Mg\ell\cos\theta_0.$$

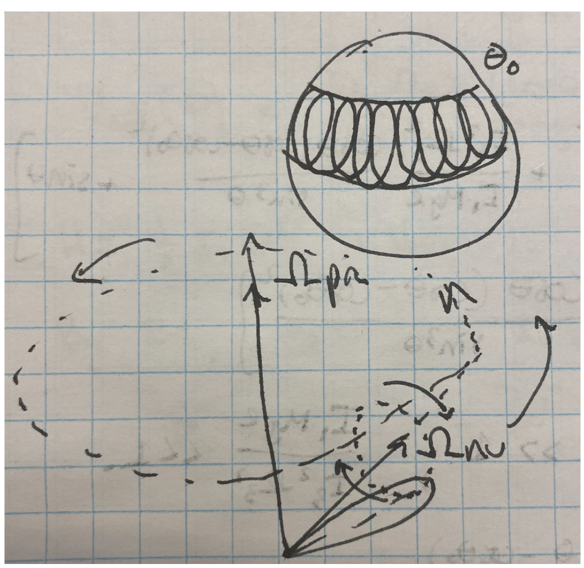
Now we cannot necessarily invoke a Taylor expansion around θ or θ_0, since neither of those quanties are neglibly small. However, when θ_0 is *close* to the minimum of $V_{\text{eff}}(\theta)$, then $\theta - \theta_0 \approx 0$, which gives a small approximation parameter (I will call it ε). Taylor expanding the effective potential to the second order around $\varepsilon = \theta - \theta_0$, we have

$$
\begin{aligned}
V_{\text{eff}}(\theta_0 + \underbrace{\theta - \theta_0}_{\equiv \varepsilon}) &\approx V_{\text{eff}}(\theta_0) + \varepsilon V'_{\text{eff}}(\theta_0) + \frac{1}{2}\varepsilon^2 V''_{\text{eff}}(\theta_0) \\
&= Mg\ell \cos\theta_0 - \varepsilon Mg\ell \sin\theta + \frac{1}{2}\varepsilon^2 \left(\frac{I_3^2 \Omega_3^2}{I_1} - Mg\ell \cos\theta_0 \right) \\
&= Mg\ell \cos\theta_0 - \varepsilon Mg\ell \sin\theta + \frac{1}{2}\varepsilon^2 \frac{I_3^2\Omega_3^2}{I_1} \left(1 - \overset{\approx 0}{\cancel{\frac{Mg\ell I_1}{I_3^2\Omega_3^2}}} \cos\theta_0 \right) \\
&= \frac{I_3^2\Omega_3^2}{I_1} \left(\overset{\approx 0}{\cancel{\frac{Mg\ell I_1}{I_3^2\Omega_3^2}}} (\cos\theta_0 - \varepsilon \sin\theta_0) + \frac{1}{2}\varepsilon^2 \right) \\
&= \frac{I_3^2\Omega_3^2}{I_1} \frac{\varepsilon^2}{2}.
\end{aligned}
$$

Taking the derivative of this approxmation (as funny as it seems) implies that $V'_{\text{eff}}(\theta) \approx \frac{I_3^2\Omega_3^2}{I_1}(\theta - \theta_0)$, which shows that when $\theta - \theta_0 \ll 1$, $V'_{\text{eff}}(\theta) \approx 0$, hence θ_0 is close to a minimum of the effective potential. Using this same approximation in the equation of motion for nutation, taking θ_0 as arbitrary gives the relation

$$
I_1 \ddot{\theta} = -\frac{d}{d\theta} \left[\frac{I_3^2\Omega_3^2}{I_1} \frac{\theta^2}{2} \right],
$$

which directly implies that $\ddot{\theta} = -\frac{I_3^2\Omega_3^2}{I_1^2}\theta$. Equivalent to the motion of a pendulum undergoing small oscillations, then the top will nutate along the generalized coordinate θ with frequency $\omega \simeq \frac{I_3\Omega_3}{I_1}$. Initially, the top is released with precessions frequency $\dot{\varphi}$ and polar angle θ_0, however as the top precesses, gravity exerts a torque on the top. The motion of the top is then a nutation with a changing azimuthal velocity:



PROBLEM 2. Find the condition for the rotation of a top about a vertical axis to be stable.

For a top to rotate purely about its vertical axis, then the $\hat{x}_3$ axis is aligned with the z axis, which implies that $M_3 = M_z$. A stable orbit occurs when the top does not oscillate between two angles θ_{min} and θ_{max}, which means that θ is a minimum of $U_{\text{eff}}(\theta)$. Taking the effective potential in terms of M_3 and M_z from **Q1**, then

$$U_{\text{eff}}(\theta) = \frac{M_z^2(1-\cos\theta)^2}{2I_1\sin^2\theta} - Mg\ell(1-\cos\theta).$$

For $\theta \ll 1$, we can Taylor expand about $\theta = 0$ to find a condition for stable rotations:

$$\begin{aligned}
U_{\text{eff}}(\theta_0) &\approx \frac{M_z^2(\left(1 - 1 + \frac{\theta^2}{2} - \text{HOT}\right)^2}{\theta^2} - Mg\ell\left(1 - 1 + \frac{\theta^2}{2}\right) \\
&= \frac{M_z^2}{2I_1}\frac{\theta^4}{4}\frac{1}{\theta^2} - \frac{1}{2}Mg\ell\theta^2 \\
&= \left(\frac{M_z^2}{8I_1} - \frac{1}{2}Mg\ell\right)\theta^2.
\end{aligned}$$

Taking $U'_{\text{eff}}(\theta) = 0$ implies that $\theta = 0$ gives a stable orbit, however the leading coefficient must be greater than zero (or else $U_{\text{eff}}(\theta) = 0$ for all θ), so

$$\frac{M_z^2}{8I_1} - \frac{1}{2}Mg\ell > 0.$$

Since $M_z = M_3 = I_3\Omega_3$, then this rearranges to give the condition for stability, which is

$$\boxed{\Omega_3^2 > \frac{4I_1 Mg\ell}{I_3^2}.}$$

3. The Lagrangian for the heavy symmetric top is

$$L = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi} \cos \theta \right)^2 - Mg\ell \cos \theta.$$

Obtain the momenta p_θ, p_ϕ and p_ψ, and the Hamiltonian $H(\theta, \phi, \psi, p_\theta, p_\phi, p_\psi)$. Derive Hamilton's equations.

In this problem, I will use the given expressions for the momenta p_φ and p_ψ as in the text for the symmetric top, pinned a distance ℓ from the center of mass. Since $\mathcal{L}$ is independent of φ and ψ, then $p_\varphi = \dfrac{\partial \mathcal{L}}{\partial \dot{\varphi}}$ and $p_\psi = \dfrac{\partial \mathcal{L}}{\partial \dot{\psi}}$ are conserved. The momenta are given by Equations (1) and (2) on page 112 of Landau and Lifshitz, as well as $p_\theta = \dfrac{\partial \mathcal{L}}{\partial \dot{\theta}}$:

$$\begin{aligned}
p_\theta &= I_1 \dot{\theta} \\
p_\varphi &= (I_1 \sin^2 \theta + I_3 \cos^2 \theta) \dot{\varphi} + I_3 \dot{\psi} \cos \theta \equiv M_z \\
p_\psi &= I_3 (\dot{\psi} + \dot{\varphi} \cos \theta) \equiv M_3.
\end{aligned}$$

Rearranging, we can solve for $\dot{\theta}$, $\dot{\varphi}$ and $\dot{\psi}$, obtaining

$$\begin{aligned}
\dot{\theta} &= \frac{p_\theta}{I_1} \\
\dot{\varphi} &= \frac{p_\varphi - p_\psi \cos \theta}{I_1 \sin^2 \theta} \\
\dot{\psi} &= \frac{p_\psi}{I_3} - \cos \theta \frac{p_\varphi - p_\psi \cos \theta}{I_1 \sin^2 \theta}.
\end{aligned}$$

The Hamiltonian is given by the total energy of the system, in this case H is

$$H = \frac{1}{2} I_1 (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) + \frac{1}{2} I_3 (\dot{\psi} + \dot{\varphi} \cos \theta)^2 + Mg\ell \cos \theta.$$

Re-writing H in terms of the previously derived quantites $\dot{\theta}$, $\dot{\varphi}$ and $\dot{\psi}$, we have that

$$H = \frac{p_\theta^2}{2I_1} + \frac{p_\psi^2}{2I_3} + \frac{(p_\varphi - p_\psi \cos \theta)^2}{2I_1 \sin^2 \theta} + Mg\ell \cos \theta.$$

I can now proceed by applying Hamilton's equations, given by $\dot{q}_i = \dfrac{\partial H}{\partial p_i}$, and $\dot{p}_i = -\dfrac{\partial H}{\partial q_i}$. For the generalized coordinate θ with corresponding momenta p_θ,

$$\begin{aligned}
\dot{\theta} = \frac{\partial H}{\partial p_\theta} &= \frac{p_\theta}{I_1} \\
\dot{p}_\theta = -\frac{\partial H}{\partial \theta} &= -\frac{\partial}{\partial \theta} \left[\frac{1}{2} I_1 (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) + \frac{1}{2} I_3 (\dot{\psi} + \dot{\varphi} \cos \theta)^2 + Mg\ell \cos \theta \right] \\
&= -I_1 \dot{\varphi}^2 \sin \theta \cos \theta + I_3 (\dot{\psi} + \dot{\varphi} \cos \theta) \sin \theta + Mg\ell \sin \theta \\
&= \sin \theta \left[-I_1 \dot{\varphi}^2 \cos \theta + I_3 (\dot{\psi} + \dot{\varphi} \cos \theta) + Mg\ell \right]
\end{aligned}$$

$$= \sin\theta \left[\frac{p_\psi(p_\varphi - p_\psi \cos\theta)}{I_1 \sin^2\theta} + Mg\ell - I_1 \cos\theta \frac{p_\varphi - p_\psi \cos\theta}{I_1 \sin^2\theta} \right].$$

For generalized coordinate and momenta φ, p_φ, Hamilton's equations give

$$\begin{aligned}
\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} &= \frac{2(p_\varphi - p_\psi \cos\theta)}{2I_1 \sin^2\theta} \\
&= \frac{p_\varphi - p_\psi \cos\theta}{I_1 \sin^2\theta} \\
\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} &= 0.
\end{aligned}$$

Lastly, the generalized coordinate and momenta ψ, p_ψ, we have

$$\begin{aligned}
\dot{\psi} = \frac{\partial H}{\partial p_\psi} &= \frac{\partial}{\partial p_\psi} \left[\frac{p_\theta^2}{2I_1} + \frac{p_\psi^2}{2I_3} + \frac{(p_\varphi - p_\psi \cos\theta)^2}{2I_1 \sin^2\theta} + Mg\ell \cos\theta \right] \\
&= 0 + \frac{p_\psi}{I_3} + \frac{2(p_\varphi - p_\psi \cos\theta)\cos\theta}{2I_1 \sin^2\theta} + 0 \\
&= \frac{p_\psi}{I_3} - \cos\theta \frac{p_\varphi - p_\psi \cos\theta}{I_1 \sin^2\theta} \\
\dot{p}_\psi = -\frac{\partial H}{\partial \psi} &= 0.
\end{aligned}$$

We find that Hamilton's equations yields the same expressions as previously defined and calculated, as required.

4. Prove that the following transformations are canonical:

   (a) $P = \frac{1}{2}(p^2 + q^2)$ and $Q = \tan^{-1}(q/p)$.

   (b) $P = q^{-1}$ and $Q = pq^2$.

   (c) $q_1 = Q_1 \cos\lambda + P_2 \sin\lambda$, $q_2 = Q_2 \cos\lambda + P_1 \sin\lambda$, $p_1 = -Q_2 \sin\lambda + P_1 \cos\lambda$, $p_2 = -Q_1 \sin\lambda + P_2 \cos\lambda$ for any constant λ.

(a) A transformation is canonical if is leaves the matrix equation $\mathcal{J} J \mathcal{J}^T = J$ invariant, where $\mathcal{J} = \begin{pmatrix} \partial_q Q & \partial_p Q \\ \partial_q P & \partial_p P \end{pmatrix}$ is the Jacobian matrix of the transformation and J is the inverse identity $J = \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0 \end{pmatrix}$.

Now, for $P = \dfrac{1}{2}(q^2 + p^2)$ and $Q = \arctan\left(\dfrac{q}{p}\right)$. The partial derivatives of each are

$$\partial_q Q = \frac{p}{q^2 + p^2} \qquad \partial_p Q = -\frac{q}{q^2 + p^2}$$

$$\partial_q P = q \qquad \partial_p P = p.$$

Then, by matrix multiplication,

$$\begin{aligned}
\mathcal{J} J \mathcal{J}^T &= \begin{pmatrix} \dfrac{p}{q^2+p^2} & -\dfrac{q}{q^2+p^2} \\ q & p \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{p}{q^2+p^2} & q \\ -\dfrac{q}{q^2+p^2} & p \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{q}{q^2+p^2} & \dfrac{p}{q^2+p^2} \\ -p & q \end{pmatrix} \begin{pmatrix} \dfrac{p}{q^2+p^2} & q \\ -\dfrac{q}{q^2+p^2} & p \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{qp - pq}{q^2+p^2} & \dfrac{q^2+p^2}{q^2+p^2} \\ \dfrac{-q^2 - p^2}{q^2+p^2} & -pq + qp \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv J,
\end{aligned}$$

and therefore this transformation is canonical.

(4b) For this transformation, we have that

$$\mathcal{J} = \begin{pmatrix} 2pq & q^2 \\ -\dfrac{1}{q^2} & 0 \end{pmatrix} \qquad \text{and again,} \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then as before, matrix multiplication follows:

$$\begin{aligned}
\mathcal{J}J\mathcal{J}^T &= \begin{pmatrix} 2pq & q^2 \\ q^{-2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2pq & q^{-2} \\ q^2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -q^2 & 2pq \\ 0 & -q^{-2} \end{pmatrix} \begin{pmatrix} 2pq & q^{-2} \\ q^2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -2qp + 2qp & -q^2(-q^{-2}) \\ -q^{-2}q^2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix} \equiv J.
\end{aligned}$$

(4c) Lastly, if a transformation is canonical, it's inverse transformation will also be canonical. The transformation is given by

$$Q_1 = cq_1 + sp_2 \qquad\qquad Q_2 = cq_2 + sp_1$$
$$P_1 = -sq_2 + cp_1 \qquad\qquad P_2 = -sq_1 + cp_2.$$

The Jacobian of the transformation is then

$$\mathcal{J} = \begin{pmatrix} \frac{\partial Q_1}{\partial q_1} & \frac{\partial Q_1}{\partial q_2} & \frac{\partial Q_1}{\partial p_1} & \frac{\partial Q_1}{\partial p_2} \\ \frac{\partial Q_2}{\partial q_1} & \frac{\partial Q_2}{\partial q_2} & \frac{\partial Q_2}{\partial p_1} & \frac{\partial Q_2}{\partial p_2} \\ \frac{\partial P_1}{\partial q_1} & \frac{\partial P_1}{\partial q_2} & \frac{\partial P_1}{\partial p_1} & \frac{\partial P_1}{\partial p_2} \\ \frac{\partial P_2}{\partial q_1} & \frac{\partial P_2}{\partial q_2} & \frac{\partial P_2}{\partial p_1} & \frac{\partial P_2}{\partial p_2} \end{pmatrix} = \begin{pmatrix} c & 0 & 0 & s \\ 0 & c & s & 0 \\ 0 & -s & c & 0 \\ -s & 0 & 0 & c \end{pmatrix}.$$

The matrix J is then given by the 2×2 identity matrices:

$$J = \begin{pmatrix} 0 & \mathbb{1}_{2\times 2} \\ -\mathbb{1}_{2\times 2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

For the last time, matrix multiplication follows:

$$\begin{aligned}
\mathcal{J}J\mathcal{J}^T &= \begin{pmatrix} c & 0 & 0 & s \\ 0 & c & s & 0 \\ 0 & -s & c & 0 \\ -s & 0 & 0 & c \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} c & 0 & 0 & -s \\ 0 & c & -s & 0 \\ 0 & s & c & 0 \\ s & 0 & 0 & c \end{pmatrix} \\
&= \begin{pmatrix} c & 0 & 0 & s \\ 0 & c & s & 0 \\ 0 & -s & c & 0 \\ -s & 0 & 0 & c \end{pmatrix} \begin{pmatrix} 0 & s & c & 0 \\ s & 0 & 0 & c \\ -c & 0 & 0 & s \\ 0 & -c & s & 0 \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} 0 & 0 & s^2+c^2 & 0 \\ 0 & 0 & 0 & s^2+c^2 \\ -s^2-c^2 & 0 & 0 & 0 \\ 0 & -s^2-c^2 & 0 & 0 \end{pmatrix} \equiv J,$$

which follows from the pythagorean trigonometric identity $\sin^2\lambda + \cos^2\lambda = 1$. Therefore this transformation is canonical, so it's inverse transformation will also be canonical.

5. * A rigid lamina (i.e. a 2 dimensional object) has principal moments of inertia about the centre of mass of

$$I_1 = (\mu^2 - 1),\ I_2 = (\mu^2 + 1),\ I_3 = 2\mu^2.$$

(a) Show, using Euler's equations, that in the body-fixed frame, the component of the angular velocity in the plane of the lamina (i.e. $\sqrt{\Omega_1^2 + \Omega_2^2}$) is constant in time.

(b) Choose the initial angular velocity to be $\vec{\Omega} = \mu N \hat{x}_1 + N\hat{x}_3$. Define $\tan\alpha = \Omega_2/\Omega_1$, which is the angle the component of Ω in the plane of the lamina makes with $\hat{x}_1$. Show that it satisfies

$$\dot{\alpha} = \dot{\Omega}_3$$

and from this show that

$$\ddot{\alpha} = -\frac{1}{\mu^2}\left(\Omega_1^2 + \Omega_2^2\right)\sin\alpha\cos\alpha = -N^2\cos\alpha\sin\alpha.$$

Show that the solution to the motion is

$$\vec{\Omega}(t) = \mu N(\cosh Nt)^{-1}\hat{x}_1 + \mu N\tanh Nt\hat{x}_2 + N(\cosh Nt)^{-1}\hat{x}_3.$$

(Note: It is enough to check that this is the solution; you do not need to solve the differential equation).

(a) To recall, Euler's equations are given by

$$\begin{aligned}
I_1\dot{\Omega}_1 + \Omega_2\Omega_3(I_3 - I_2) &= 0\\
I_2\dot{\Omega}_2 + \Omega_1\Omega_3(I_1 - I_3) &= 0\\
I_3\dot{\Omega}_3 + \Omega_1\Omega_2(I_2 - I_1) &= 0.
\end{aligned}$$

The principle axes of inertia through the center of mass of the lamina are given to be $I_1 = \mu^2 - 1$, $I_2 = \mu^2 + 1$ and $I_3 = 2\mu^2$. Notice subtle symmetry, and how

$$I_3 - I_2 = I_1, \quad I_1 - I_3 = -I_2 \quad \text{and} \quad I_2 - I_1 = 2.$$

We can easily incorporate these relations into Euler's equations above:

$$I_1\dot{\Omega}_1 + I_1\Omega_2\Omega_3 = 0 \implies \dot{\Omega}_1 = -\Omega_2\Omega_3 \tag{5.1}$$

$$I_2\dot{\Omega}_2 - I_2\Omega_1\Omega_3 = 0 \implies \dot{\Omega}_2 = \Omega_1\Omega_3 \tag{5.2}$$

$$I_3\dot{\Omega}_3 + 2\Omega_1\Omega_2 = 0 \implies \dot{\Omega}_3 = -\frac{2}{I_3}\Omega_1\Omega_2. \tag{5.3}$$

Now, to show that the component of $\vec{\Omega}$ in the plane of the lamina, that is $\sqrt{\Omega_1^2 + \Omega_2^2}$, is constant in time, then $\frac{d}{dt}\left[\sqrt{\Omega_1^2 + \Omega_2^2}\right] = 0$. We have that

$$\frac{d}{dt}\left[\sqrt{\Omega_1^2 + \Omega_2^2}\right] = \frac{1}{2\sqrt{\Omega_1^2 + \Omega_2^2}}\frac{d}{dt}[\Omega_1^2 + \Omega_2^2]$$

$$
\begin{aligned}
&= \frac{2\Omega_1\dot{\Omega}_1 + 2\Omega_2\dot{\Omega}_2}{2\sqrt{\Omega_1^2+\Omega_2^2}} = \frac{\Omega_1 \overbrace{(-\Omega_2\Omega_3)}^{\text{by (5.1)}} + \Omega_2 \overbrace{(\Omega_1\Omega_3)}^{\text{by (5.2)}}}{2\sqrt{\Omega_1^2+\Omega_2^2}} \\
&= \frac{-\Omega_1\Omega_2\Omega_3 + \Omega_1\Omega_2\Omega_3}{\sqrt{\Omega_1^2+\Omega_2^2}} \\
&= 0,
\end{aligned}
$$

as desired. Therefore the component of angular momentum in the plane of the lamina is constant in time.

(5b) Since $\tan\alpha = \dfrac{\Omega_2}{\Omega_1}$, we can solve for α then proceed by taking time derivatives:

$$
\begin{aligned}
\alpha = \arctan\left(\frac{\Omega_2}{\Omega_1}\right) \implies \dot{\alpha} &= \underbrace{\frac{1}{\Omega_2^2/\Omega_1^2+1}\cdot\frac{d}{dt}\left[\frac{\Omega_2}{\Omega_1}\right]}_{\text{chain rule}} \\
&= \frac{1}{\Omega_2^2/\Omega_1^2+1}\cdot\underbrace{\frac{\Omega_1\dot{\Omega}_2 - \dot{\Omega}_1\Omega_2}{\Omega_1^2}}_{\text{quotient rule}} \\
&= \frac{\Omega_1\dot{\Omega}_2 - \dot{\Omega}_1\Omega_2}{\Omega_1^2+\Omega_2^2} = \frac{\Omega_1 \overbrace{(\Omega_1\Omega_3)}^{\text{by (5.2)}} - \overbrace{(\Omega_2\Omega_3)}^{\text{by (5.1)}}\Omega_2}{\Omega_1^2+\Omega_2^2} \\
&= \frac{\Omega_3(\Omega_1^2+\Omega_2^2)}{(\Omega_1^2+\Omega_2^2)} = \Omega_3.
\end{aligned}
$$

Taking a second time derivative simply yields that $\ddot{\alpha} = \dot{\Omega}_3$, which is what I wanted to show. Now, from **(5a)**, we found that the component of $\vec{\Omega}$ which lies in the plane of the lamina is given by $\sqrt{\Omega_1^2+\Omega_2^2}$ and is constant in time. If the angle between $\sqrt{\Omega_1^2+\Omega_2^2}$ and Ω_1 is α (since $\tan\alpha = \dfrac{\Omega_2}{\Omega_1}$), then respectively

$$
\Omega_1 = \sqrt{\Omega_1^2+\Omega_2^2}\cos\alpha \qquad \text{and} \qquad \Omega_2 = \sqrt{\Omega_1^2+\Omega_2^2}\sin\alpha. \tag{5.4}
$$

By Equation (5.3), we have that $\dot{\Omega}_3 = -\dfrac{2}{I_3}\Omega_1\Omega_2 = -\dfrac{1}{\mu^2}\Omega_1\Omega_2$. Therefore since $\ddot{\alpha} = \dot{\Omega}_3$, then by Equation (5.4),

$$
\begin{aligned}
\dot{\Omega}_3 &= -\frac{1}{\mu^2}\left(\sqrt{\Omega_1^2+\Omega_2^2}\cos\alpha\right)\left(\sqrt{\Omega_1^2+\Omega_2^2}\sin\alpha\right) \\
&= -\frac{1}{\mu^2}(\Omega_1^2+\Omega_2^2)\cos\alpha\sin\alpha.
\end{aligned}
$$

Initially, if $\Omega_1 = \mu N$ and $\Omega_2 = 0$, then $\ddot{\alpha} = -\dfrac{1}{\mu^2}(\mu^2N^2+0)\cos\alpha\sin\alpha = -N^2\cos\alpha\sin\alpha$. To show that

$$
\vec{\Omega} = \mu N(\cosh Nt)^{-1}\hat{x}_1 + \mu N\tanh Nt\hat{x}_2 + N(\cosh Nt)^{-1}\hat{x}_3
$$

is a solution to the differential equation, it suffices to check whether Euler's equations are satisfied, since the ODE $\ddot{\alpha} = \dot{\Omega}_3$ is a product of those equations. The components are given by

$$\Omega_1 = \mu N[\cosh Nt]^{-1}, \quad \Omega_2 = \mu N[\tanh Nt], \quad \Omega_3 = N[\cosh Nt]^{-1}.$$

Then

$$\begin{aligned}
\dot{\Omega}_1 &= -\mu N[\cosh Nt]^{-2}\frac{d}{dt}[\cosh Nt] \\
&= -\mu N^2[\cosh Nt]^{-2}\sinh Nt.
\end{aligned}$$

$$\begin{aligned}
\dot{\Omega}_2 &= \mu N^2[1-\tanh^2 Nt] \\
&= \mu N^2 \operatorname{sech}^2 Nt \\
&= \mu N^2[\cosh Nt]^{-2} \\
\dot{\Omega}_3 &= -N^2[\cosh Nt]^{-2}\sinh Nt.
\end{aligned}$$

Applying Euler's Equations, we have that

$$\begin{aligned}
(5.1): \qquad -\mu N^2[\cosh Nt]^{-2}\sinh Nt &= -(\mu N \tanh Nt)(N[\cosh Nt]^{-1}) \\
&= -\mu N^2[\cosh Nt]^{-2}\sinh Nt
\end{aligned}$$

$$\begin{aligned}
(5.2): \qquad \mu N^2[\cosh Nt]^{-2} &= (\mu N[\cosh Nt]^{-1})(N[\cosh Nt]^{-1}) \\
&= \mu N^2[\cosh Nt]^{-2}
\end{aligned}$$

$$\begin{aligned}
(5.3): \qquad -N^2[\cosh Nt]^{-2}\sinh Nt &= -\frac{1}{\mu^2}(\mu N[\cosh Nt]^{-1})(\mu N[\cosh Nt]^{-1}\sinh Nt) \\
&= -N^2[\cosh Nt]^{-2}\sinh Nt.
\end{aligned}$$

6. * A particle with mass m, position $\vec{r}$ and momentum $\vec{p}$ has angular momentum $\vec{M} = \vec{r} \times \vec{p}$.

(a) Evaluate $\{x_j, M_k\}$, $\{p_j, M_k\}$, $\{M_j, M_k\}$ and $\{M_i, \vec{M}^2\}$.

(b) Consider a particle moving of mass m moving in a central potential $V(r) = -mK/r$. Recall the Runge-Lenz vector

$$\vec{A} = \vec{p} \times \vec{M} - mK\hat{r}.$$

Show that

$$\{M_i, A_j\} = \epsilon_{ijk} A_k, \ \{A_i, A_j\} = -\frac{2H}{m}\epsilon_{ijk} M_k$$

where

$$\epsilon_{ijk} = \begin{cases} +1, & \text{if } (i,j,k) \text{ is an even permutation of (1,2,3)} \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of (1,2,3)} \\ 0 & \text{otherwise} \end{cases}.$$

Prove using Poisson brackets that $\vec{A}$ is conserved. (Also note that $\vec{A}$, $\vec{M}$ and H form a closed algebra under the Poisson bracket.)

(a) The Poisson bracket is given by $\{f, g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}$. The angular momentum is defined as $\vec{M} = \vec{r} \times \vec{p}$, however we are only wanting to isolate the $j - th$ component of this cross product. Using the Levi-Civita permutation tensor, we can rewrite $M_j = (\vec{r} \times \vec{p})_j = e_{jkl} x_k p_l$. This will simplify the Poisson bracket calculations. If any of the indices i, j, k are equal, $e_{jkl} = 0$. With delta functions, this implies that $e_{ijk}\delta_{jk} = 0$ always. Furthermore, 'swapping' any of the indices changes the sign of the tensor: $e_{ijk} = -e_{ikj} = \ldots$. Using Einstein summation, we have

$$\begin{aligned} \{x_j, M_k\} &= \frac{\partial x_j}{\partial q_i}\frac{\partial M_k}{\partial p_i} - \cancelto{0}{\frac{\partial x_j}{\partial p_i}}\frac{\partial M_k}{\partial q_i} = \delta_{ij}\frac{\partial}{\partial p_i}[e_{kab} x_a p_b] \\ &= \delta_{ij} e_{kab} x_a p_b \delta_{ib} = e_{kaj} x_a = e_{ajk} x_a \end{aligned}$$

$$\begin{aligned} \{p_j, M_k\} &= \cancelto{0}{\frac{\partial p_j}{\partial q)u}}\frac{\partial M_K}{\partial p_i} - \frac{\partial p_j}{\partial p_i}\frac{\partial M_k}{\partial q_i} = -\delta_{ij}\frac{\partial}{\partial q_i}[e_{kab} x_a p_b] \\ &= -\delta_{ij} e_{kab} p_b \delta_{ia} = -e_{kjb} p_b = e_{bjk} p_b \end{aligned}$$

$$\begin{aligned} \{M_j, M_k\} &= \frac{\partial M_j}{\partial q_i}\frac{\partial M_k}{\partial p_i} - \frac{\partial M_k}{\partial q_i}\frac{\partial M_j}{\partial p_i} = \frac{\partial}{\partial q_i}[e_{jkl} x_k p_l]\frac{\partial}{\partial p_i}[e_{klj} x_l p_i] - \frac{\partial}{\partial p_i}[e_{jkl} x_k p_l]\frac{\partial}{\partial q_i}[e_{klj} x_l p_i] \\ &= (e_{ikj}\delta_{ik} p_l)(e_{klj}\delta_{ij} x_l) - (e_{klj}\delta_{il} p_j)(e_{jkl}\delta_{il} x_k) \\ &= \cancelto{0}{e_{ijk}\delta_{jk} p_l x_l} - e_{jkl}\delta_i l x_k p_j \\ &= e_{ikj} x_k p_j = M_i \end{aligned}$$

The magnitude of the angular momentum squared can also be expressed using the Levi-Civita tensor. Since $\vec{M}^2 = M_x^2 + M_y^2 + M_z^2$, we can sum over each of the components squared:

$\vec{M}^2 = (e_{nkl}x_k p_l)^2$, summing over n. Using this fact,

$$
\begin{aligned}
\{M_j, \vec{M}^2\} = \frac{\partial M_j}{\partial q_i}\frac{\partial \vec{M}^2}{\partial p_i} - \frac{\partial M_j}{\partial p_i}\frac{\partial \vec{M}^2}{\partial q_i} &= \frac{\partial}{\partial q_i}[e_{jkl}x_k p_l]\frac{\partial}{\partial p_i}[(e_{nkl}x_k p_l)^2] - \frac{\partial}{\partial p_i}[e_{jkl}x_k p_l]\frac{\partial}{\partial q_i}[(e_{nkl}x_k p_l)^2] \\
&= (e_{jkl}\delta_{ik}p_l)(2e_{nkl}x_k p_l x_k \delta_{il}) - (2e_{nkl}x_k p_l^2 \delta_{il})(e_{jkl}x_k\delta_{ik}) \\
&= 2e_{jkl}e_{nkl}\delta_{lk}x_k^2 p_l^2 - 2e_{nkl}e_{jkl}\delta_{kl}x_k^2 p_l^2 \\
&= 0,
\end{aligned}
$$

which follows by the Levi-Civita tensor and the delta functions since $i = k$ or $i = l$, and the terms cancel. Therefore

$$\boxed{\{x_j, M_k\} = e_{ajk}x_a, \quad \{p_j, M_k\} = e_{bjk}p_b, \quad \{M_j, M_k\} = M_i, \quad \{M_j, \vec{M}^2\} = 0.}$$

(6b) For this problem, I will begin by determining a singular component of $\vec{A}$, which then I can determine $\frac{\partial A_j}{\partial q_i}$ and $\frac{\partial A_j}{\partial p_i}$. Note that

$$\vec{A} = \vec{p} \times \vec{M} - mK\hat{r} = \vec{p} \times (\vec{r} \times \vec{p}) - mK\frac{\vec{r}}{r}.$$

By applying the triple product identity to the $\vec{p}\times(\vec{r}\times\vec{p})$ term, we have that $\vec{A} = p^2\vec{r} - (\vec{p}\cdot\vec{r})\vec{p} - mK\frac{\vec{r}}{r}$. This easily gives us each of the components of $\vec{A}$:

$$A_j = x_j p^2 - p_j(\vec{p}\cdot\vec{r}) - mK\frac{x_j}{r},$$

where I have relabelled $r_j \equiv x_j$ to avoid any confusion between the magnitude r. Note that the magnitudes r and p each still depend on the components. Then

$$
\begin{aligned}
\frac{\partial A_j}{\partial q_i} &= \delta_{ij}p^2 - \frac{\partial}{\partial q_i}(\vec{p}\cdot\vec{r})p_j - \frac{mK}{r}\delta_{ij} - mKx_j\frac{\partial}{\partial q_i}\frac{1}{r} \\
&= \delta_{ij}p^2 - p_i p_j - \frac{mK}{r}\delta_{ij} + \frac{mKx_i x_j}{r^3}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial A_j}{\partial p_i} &= x_j\frac{\partial}{\partial p_i}p^2 - \frac{\partial}{\partial p_i}(\vec{p}\cdot\vec{r})p_j - (\vec{p}\cdot\vec{r})\delta_{ij} \\
&= 2p_i x_j - x_i p_j - (\vec{p}\cdot\vec{r})\delta_{ij}.
\end{aligned}
$$

As before, we can write a component of angular momentum using the Levi-Civita tensor:
$M_i = e_{inm}x_n p_m$. I wish to determine the Poisson bracket $\{M_i, A_j\}$, however for sanity's sake, I would like to keep my partial derivatives with respect to the $i-th$ component. Therefore I will reindex to instead determine $\{M_j, A_k\}$:

$$\{M_j, A_k\} = \frac{\partial M_j}{\partial q_i}\frac{\partial A_k}{\partial p_i} - \frac{\partial M_j}{\partial p_i}\frac{\partial A_k}{\partial q_i}$$

$$
\begin{aligned}
&= e_{jnm}\delta_{in}p_m(2p_ix_k - x_ip_k - (\vec{p}\cdot\vec{r})\delta_{ik}) - e_{jnm}x_n\delta_{im}(\delta_{ij}p^2 - p_ip_k - \frac{mK}{r}\delta_{ij} + \frac{mKx_ix_j}{r^3}) \\
&= \underbrace{e_{jnm}p_m(2p_nx_k - x_np_k)}_{\text{by } \delta_{in}} - \underbrace{e_{jkm}p_m(\vec{p}\cdot\vec{r})}_{\text{by } \delta_{in}\delta_{ik}=\delta_{nk}} - \underbrace{e_{jnk}\left(x_np^2 - \frac{mKx_n}{r}\right)}_{\text{by } \delta_{im}\delta_{ik}=\delta_{mk}} + e_{jnm}\left(p_mp_kx_n - mK\frac{x_kx_mx_n}{r^3}\right) \\
&= e_{jnm}\left(2p_np_mx_k \cancelto{0}{- p_kp_mx_n + p_kp_mx_n} - mK\frac{x_kx_mx_n}{r^3}\right) - \underbrace{e_{jkl}p_l(\vec{p}\cdot\vec{r}) - e_{jlk}\left(x_lp^2 - mK\frac{x_l}{r}\right)}_{\text{by index change } n\to l} \\
&= 2x_k\cancelto{0}{(\vec{p}\times\vec{p})_j} - mK\frac{x_k}{r^3}\cancelto{0}{(\vec{r}\times\vec{r})_j} + e_{jkl}\left(x_lp^2 - p_l(\vec{p}\cdot\vec{r}) - mK\frac{x_l}{r}\right) \\
&= e_{jkl}A_l.
\end{aligned}
$$

Now to determine $\{A_j, A_k\}$ (I changed the indices again). The Hamiltonian is given by the total energy of the particle, in which case is $H = T + V = \dfrac{p^2}{2m} - \dfrac{K}{r}$. We have that

$$
\begin{aligned}
\{A_j, A_k\} &= \frac{\partial A_j}{\partial q_i}\frac{\partial A_k}{\partial p_i} - \frac{\partial A_k}{\partial q_i}\frac{\partial A_j}{\partial p_i} \\
&= \left(\delta_{ij}p^2 - p_ip_j - \frac{mK}{r}\delta_{ij} + \frac{mKx_ix_j}{r^3}\right)(2p_ix_k - x_ip_k - (\vec{p}\cdot\vec{r})\delta_{ik}) \\
&- \left(\delta_{ik}p^2 - p_ip_k - \frac{mK}{r}\delta_{ik} + \frac{mKx_ix_k}{r^3}\right)(2p_ix_j - x_ip_j - (\vec{p}\cdot\vec{r})\delta_{ij}).
\end{aligned}
$$

After too much algebra which I do not wish to typeset (a photo is included below), we have that

$$
\begin{aligned}
\{A_j, A_k\} &= -2m\left(\frac{p^2}{2m} - \frac{k}{r}\right)(x_kp_j - x_jp_k) \\
&= \ldots\ldots \\
&= -2mHe_{jki}M_i.
\end{aligned}
$$

$$\{A_j, A_k\} = \frac{\partial A_j}{\partial q_i}\frac{\partial A_k}{\partial p_i} - \frac{\partial A_j}{\partial p_i}\frac{\partial A_k}{\partial q_i}$$

$$= \left(\delta_{ij} p^2 - p_i p_j - \frac{mK}{r}\delta_{ij} + \frac{mK x_i x_j}{r^3}\right)\left(2p_i x_k - x_i p_k - (\vec{p}\cdot\vec{r})\delta_{ik}\right) - \left[2p_i x_j - x_i p_j - (\vec{p}\cdot\vec{r})\delta_{ij}\right] \cdot \left(\delta_{ik} p^2 - p_i p_k - \frac{mK}{r}\delta_{ik} + \frac{mK x_i x_k}{r^3}\right)$$

$$= \delta_{ij} p^2 \cdot 2p_i x_k - \delta_{ij} p^2 x_i p_k - \delta_{ij}\delta_{ik} p^2 (\vec{p}\cdot\vec{r}) - 2p_i^2 p_j x_k + x_i p_i p_j p_k + \delta_{ik} p_i p_j (\vec{p}\cdot\vec{r})$$
$$- \delta_{ij}\frac{mK}{r}\cdot 2p_i x_k + \frac{mK}{r}\delta_{ij}\cdot x_i p_k + \frac{mK}{r}(\vec{p}\cdot\vec{r})\delta_{ij}\delta_{ik} + \frac{mK}{r^3}\cdot 2x_i x_j x_k p_i - \frac{mK}{r^3}x_i^2 x_j p_k - \frac{mK}{r^3}\delta_{ik}(\vec{p}\cdot\vec{r})x_i x_j$$
$$- \Big[\delta_{ik} p^2 \cdot 2p_i x_j - \delta_{ik} p^2 x_i p_j - \delta_{ij}\delta_{ik}(\vec{p}\cdot\vec{r}) p^2 - 2p_i^2 p_k x_j + x_i p_i p_k p_j + \delta_{ij}(\vec{p}\cdot\vec{r}) p_i p_k$$
$$- \frac{mK}{r}\delta_{ik}\cdot 2p_i x_j + \frac{mK}{r}\delta_{ik} x_i p_j + \frac{mK}{r}\delta_{ij}\delta_{ik}(\vec{p}\cdot\vec{r}) + \frac{mK}{r^3}\cdot 2x_i x_k p_i x_j - \frac{mK}{r^3}x_i^2 x_k p_j - \frac{mK}{r^3}\delta_{ij} x_i x_k (\vec{p}\cdot\vec{r})\Big]$$

$$= \cancel{2p^2 p_j x_k} - p^2 x_j p_k - \cancel{\delta_{jk} p^2 (\vec{p}\cdot\vec{r})} - \cancel{2p^2 p_j x_k} + \cancel{p_j p_k (\vec{r}\cdot\vec{p})} + \cancel{p_k p_j (\vec{p}\cdot\vec{r})}$$
$$- \frac{mK}{r}\cdot 2p_j x_k + \frac{mK}{r}x_j p_k + \cancel{\frac{mK}{r}(\vec{p}\cdot\vec{r})\delta_{jk}} + \cancel{\frac{mK}{r^3}\cdot 2(\vec{r}\cdot\vec{p})x_j x_k} - \frac{mK}{r^3}r^2 x_j p_k - \cancel{\frac{mK}{r^3}(\vec{p}\cdot\vec{r})x_k x_j}$$
$$- \cancel{2p^2 p_k x_j} + p^2 x_k p_j + \cancel{\delta_{jk}(\vec{p}\cdot\vec{r})p^2} + \cancel{2p^2 p_k x_j} - \cancel{(\vec{r}\cdot\vec{p})p_k p_j} - \cancel{(\vec{p}\cdot\vec{r})p_j p_k}$$
$$+ \frac{mK}{r}\cdot 2p_k x_j - \frac{mK}{r}x_k p_j - \cancel{\frac{mK}{r}\delta_{jk}(\vec{p}\cdot\vec{r})} - \cancel{\frac{mK}{r^3}\cdot 2(\vec{p}\cdot\vec{r})x_k x_j} + \frac{mK}{r^3}r^2 x_k p_j + \cancel{\frac{mK}{r^3}x_j x_k(\vec{p}\cdot\vec{r})}$$

$$= -p^2 x_j p_k - \frac{mK}{r}3p_j x_k + \frac{mK}{r}3p_k x_j - \frac{mK}{\cancel{r^3}}\cancel{r^2}x_j p_k + \frac{mK}{\cancel{r^3}}\cancel{r^2}x_k p_j + p^2 x_k p_j$$

$$= p^2 x_k p_j - p^2 x_j p_k + \frac{mK}{r}\left[3p_k x_j - 3p_j x_k - x_j p_k + x_k p_j\right]$$

$$= p^2(x_k p_j - x_j p_k) + \frac{mK}{r}\left[2p_k x_j - 2p_j x_k\right]$$

$$= p^2(x_k p_j - x_j p_k) - \frac{2mK}{r}(x_k p_j - x_j p_k)$$

$$= \left(p^2 - \frac{2mK}{r}\right)(x_k p_j - x_j p_k)$$

$$= -2m\left(\frac{p^2}{2m} - \frac{K}{r}\right)(x_j p_k - x_k p_j)$$

$$= -2m H \epsilon_{ijk} M_i$$

$$= -2m H \epsilon_{jki} M_i .$$

I should note that summation is implied over i, so $x_i^2 = r^2$, $p_i^2 = p^2$ and $x_i p_i = \vec{r}\cdot\vec{p}$. This property was previously applied and I will apply it again to the next Poisson bracket calculation. Now if $\vec{A}$ is conserved, then $\vec{A}$ will poisson commute with the Hamiltonian, that is $\{A_k, H\} = 0$. To begin, I will find the quantities $\dfrac{\partial H}{\partial q_i}$ and $\dfrac{\partial H}{\partial p_i}$:

$$\frac{\partial H}{\partial q_i} = 0 - K\frac{\partial}{\partial q_i} = \frac{K}{r^3}x_i$$
$$\frac{\partial H}{\partial p_i} = \frac{p_i}{m}.$$

Again, because I do not wish typesetting all of this algebra, we have that $\{A_k, H\} = 0$, which holds for all components of $\vec{A}$, and therefore $\vec{A}$ is conserved.

$$\{A_k, H\} = \frac{\partial A_k}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A_k}{\partial p_i}\frac{\partial H}{\partial q_i}$$
$$= \left(\delta_{ik}p^2 - p_i p_k - \frac{mK}{r}\delta_{ik} + \frac{mK x_i x_k}{r^3}\right)\left(\frac{p_i}{m}\right) - \left(2p_i x_k - x_i p_k - \delta_{ik}(\vec{p}\cdot\vec{r})\right)\left(\frac{K x_i}{r^3}\right)$$
$$= \left[\delta_{ik}p^2 p_i - p_i^2 p_k - \frac{mK}{r}\delta_{ik}p_i + \frac{mK x_i x_k p_i}{r^3}\right]\left(\frac{1}{m}\right) - \left[\frac{2K x_i x_k p_i}{r^3} - \frac{K x_i^2 p_k}{r^3} - \frac{\delta_{ik}(\vec{p}\cdot\vec{r})K x_i}{r^3}\right]$$

By implicit summation, $x_i p_i \equiv \vec{p}\cdot\vec{r}$, $x_i^2 = r^2$ and $p_i^2 = p^2$.

$$= \left[\cancel{p^2 p_k - p^2 p_k}^{\,0} - \frac{mK p_k}{r} + \frac{mK x_k(\vec{p}\cdot\vec{r})}{r^3}\right]\frac{1}{m} - \frac{2K x_k(\vec{p}\cdot\vec{r})}{r^3} + \frac{K p_k}{r} + \frac{(\vec{p}\cdot\vec{r})K x_k}{r^3}$$
$$= -\cancel{\frac{K p_k}{r}} + \cancel{\frac{K x_k(\vec{p}\cdot\vec{r})}{r^3}} - \cancel{\frac{2K x_k(\vec{p}\cdot\vec{r})}{r^3}} + \cancel{\frac{K p_k}{r}} + \cancel{\frac{(\vec{p}\cdot\vec{r})K x_k}{r^3}}$$
$$= 0.$$

Therefore

$$\boxed{\{M_j, A_k\} = e_{jkl}A_l \quad \{A_j, A_k\} = -2mHe_{kji}M_i \quad \text{and} \quad \{A_k, H\} = 0.}$$