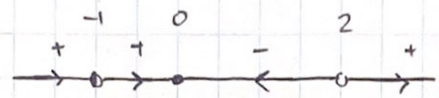
PHY 460 - Problem Set 1 DUE 10/03/2022

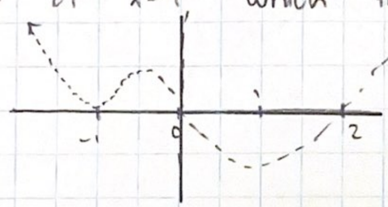
Q 2.2.8

We are given the vector field



and we wish to find a function $f(x)$ which satisfies this field, provided $\dot{x} = f(x)$. Whenever a vector arrow is moving rightward, $f(x) > 0$ and similarly $f(x) < 0$ leftward. I will proceed by finding the most simple function $f(x)$.

We have that $f(x)$ must have roots at $x = -1, 0, 2$. We have a positive 'bump' at $x=-1$ which then appears as



$f(x)$ is then $\boxed{f(x) = x(x+1)^2(x-2)}$

Q3. Model of Tumor Growth

Gompertz Law: $\dot{N} = -aN \log(bN)$.

a) To make this problem easier to interpret, I will solve the ODE for $N(t)$. We have that

$$\frac{dN}{dt} = -aN\log(bN) \implies \frac{dN}{N\log(bN)} = -a\,dt.$$

Integration yields that $\int_{N_0}^{N(t)} \frac{dN'}{N'\log(bN')} = -a(t-t_0)$ in which I will set $t_0 = 0$. A u-sub of the integrand, let $u = \log(bN')$, $du = \frac{1}{N'}dN'$

thus $\int \frac{du}{u} \to \log[\log(bN')]\Big|_{N_0}^{N(t)} = \log[\log(bN)] - \log[\log(bN_0)]$

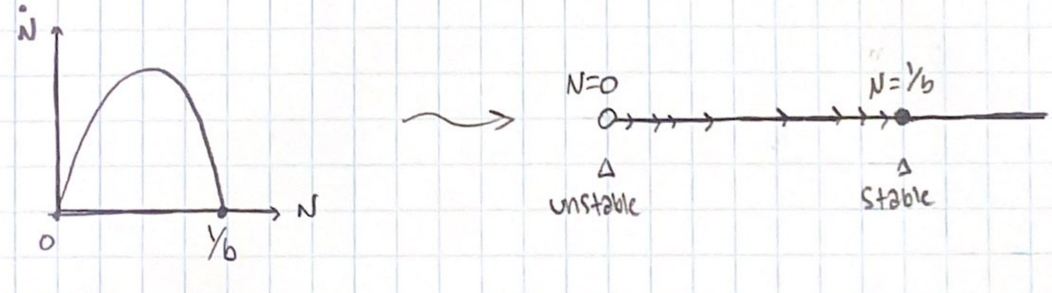
Then $e^{-at} = \frac{\log(bN)}{\log(bN_0)} \longrightarrow e^{-at}\log(bN_0) = \log(bN)$.

Exponentiating again, we find that

$$N(t) = \frac{1}{b}\exp\left[\log(bN_0)\,e^{-at}\right].$$

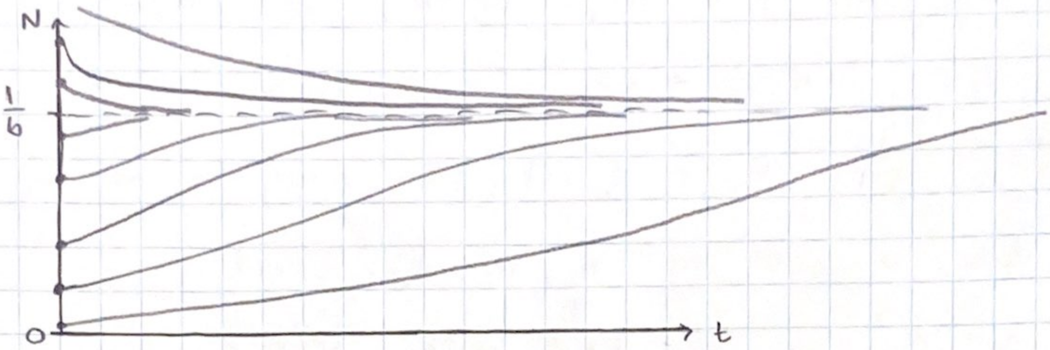
It is easy to see now that $\underline{a}$ is a growth rate or moreover, the rate of proliferation of the cells. Likewise, $\underline{b}$ is the carrying capacity, since $\lim_{t\to\infty} N(t) = \frac{1}{b}$.

b) The vector field of N appears like



Since the domain of N is limited to $[0, 1/b]$.

We can now plot $N(t)$ for varias initial values:



or something very similar to this.

c) The fixed points of the model are

$N=0$ [unstable] and

$N=\frac{1}{b}$ [Stable],

which physically makes sense, since as $t \to \infty$, the number of cells should converge to the carrying capacity, as when $t=0$, any initial perturbation (initial cell count which is not zero) shall send the model into play.

d) The stability analysis is given by $\dot{\eta} \approx \eta\, f'(N^*)$ where

$f(N) = -aN\log(bN)$.

N^* are the fixed points $N^*=0$, and $N^*=\frac{1}{b}$. The derivative of f is

$$f'(N) = -a\log(bN) - aN \cdot \frac{1}{bN} \cdot b$$

$$= -a\left[\log(bN)+1\right].$$

Now, when $N^* = 0$, $\log(bN^*)$ is undefined to $-\infty$, hence $f'(N^*=0) \to +\infty$. The characteristic time scale is then

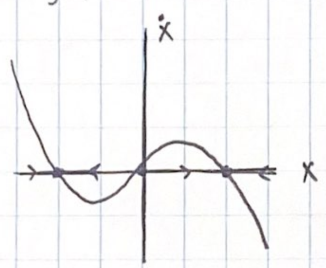
$\frac{1}{|\infty|} \simeq 0$, ie, it takes virtually no time for a tumor to grow once one cell has appeared.

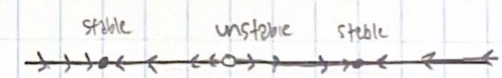
Similarly, for the fixed point $N^* = \frac{1}{b}$, the characteristic time scale is

$$\frac{1}{|f'(N^*)|} = \frac{1}{|-a[\log(b\cdot\frac{1}{b})+1]|} = \frac{1}{a}.$$

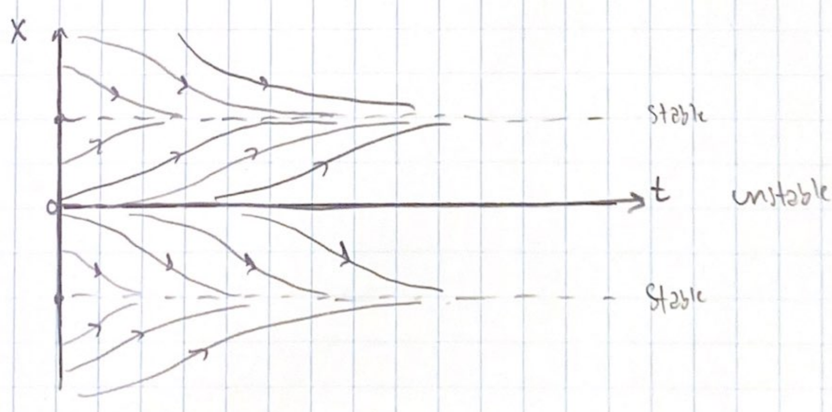
e) (i) $\dot{x} = x - x^3$. This can be factored into $\dot{x} = x(1-x)(1+x)$,

so our graph of $\dot{x}$-x looks like



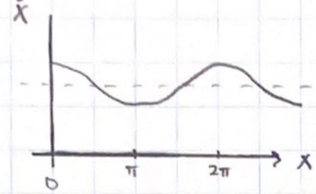
and the corresponding vector field is 

The solution can be graphed as



(ii) $\dot{x} = 1 + \frac{1}{2}\cos x$. This function has no roots, since

$$-\tfrac{1}{2} \le \tfrac{1}{2}\cos x \le \tfrac{1}{2} \implies 1 - \tfrac{1}{2} \le 1 + \tfrac{1}{2}\cos x \le \tfrac{1}{2} + 1.$$



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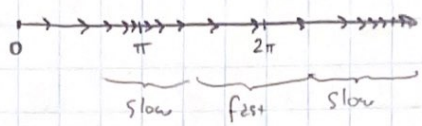
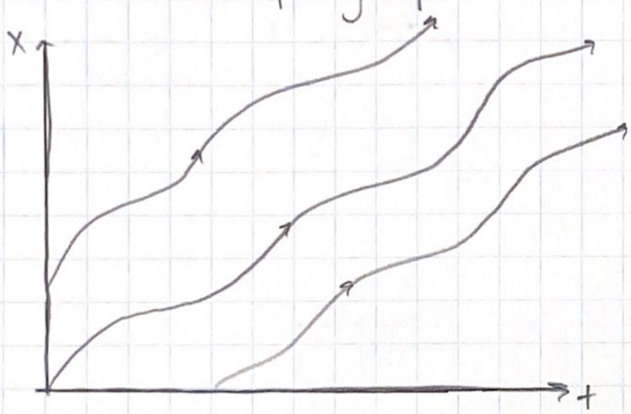


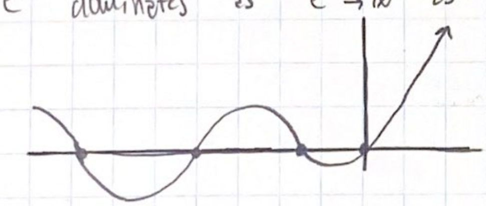
no fixed points

and the corresponding plot looks like

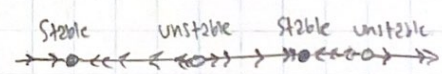


for various initial values of x and t.

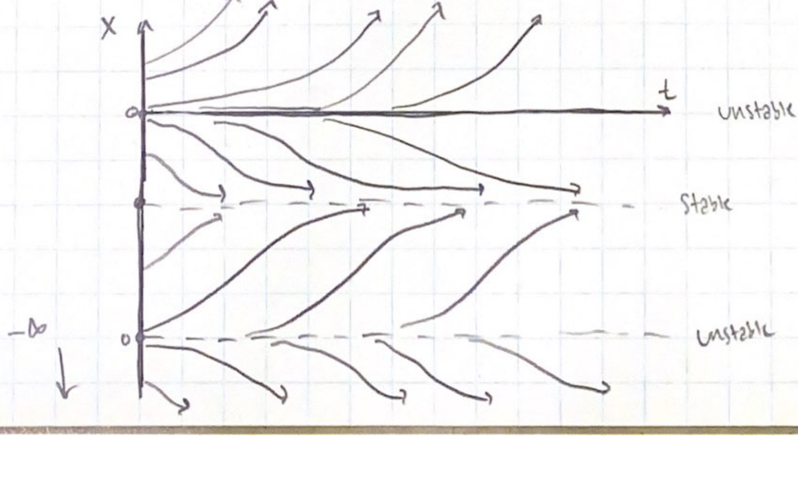
(iii) $\dot{x} = e^x - \cos x$. When $x<0$, $-\cos x$ dominates, however when $x>0$, e^x dominates as $e^x \to \infty$ as $x \to \infty$.



$\longrightarrow$



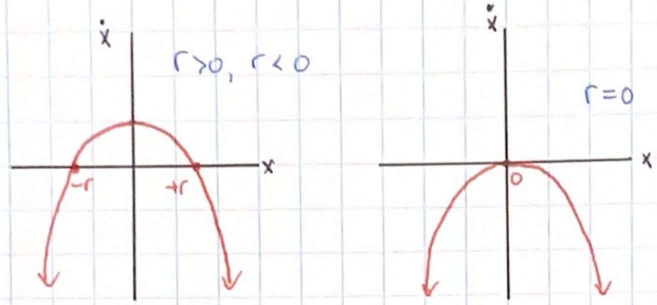
As $x \to -\infty$, the fixed points alternate as stable, unstable, stable, etc. When $x>0$, $\dot{x}$ is unbounded. The corresponding plot of $x(t)$ looks like



5. Bifurcations

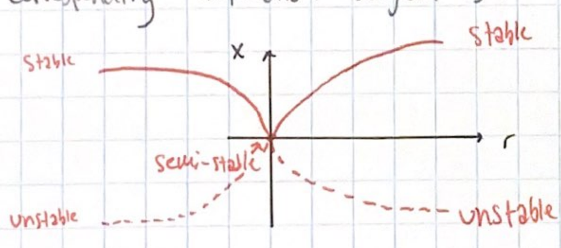
a) $\dot{x} = r^2 - x^2$. Can factor into $\dot{x} = (r-x)(r+x)$.

When $r>0$, $\dot{x}$ is the same as when $r<0$ since $r^2 \geq 0$. The roots (or fixed points) are given when $x = \pm r$.



The corresponding vector fields:

Once $r \geq 0$, a set of fixed points are created. Hence the corresponding bifurcation diagram is:



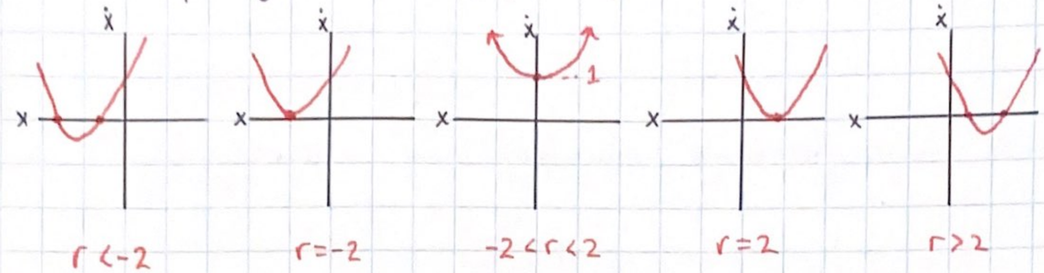
which is a <u>saddle-node bifurcation</u> at $r=0$.

c) $\dot{x} = 1 - rx + x^2$. By applying the quadratic formula, the roots are determined to be

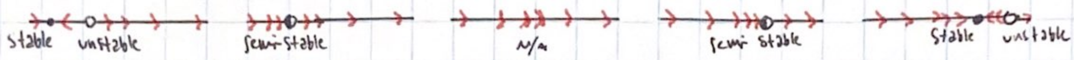
$$x = \frac{r \pm \sqrt{r^2-4}}{2}.$$

When $r>2$, or $r<-2$, we have that x has two positive and two negative real roots, respectively. When $r = \pm 2$, $x = \pm 1$ have individual single roots. When $-2 < r < 2$, $\dot{x}$ has no real roots.

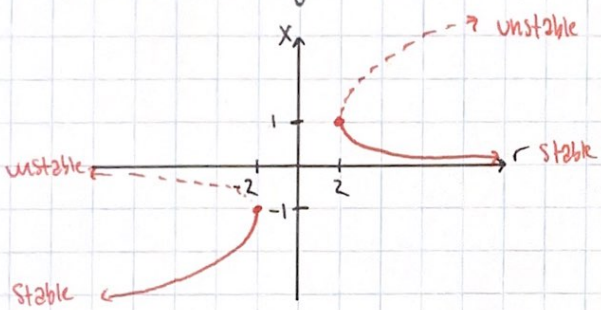
The corresponding $\dot{x}$ vs x graphs appear as:



and vector fields:

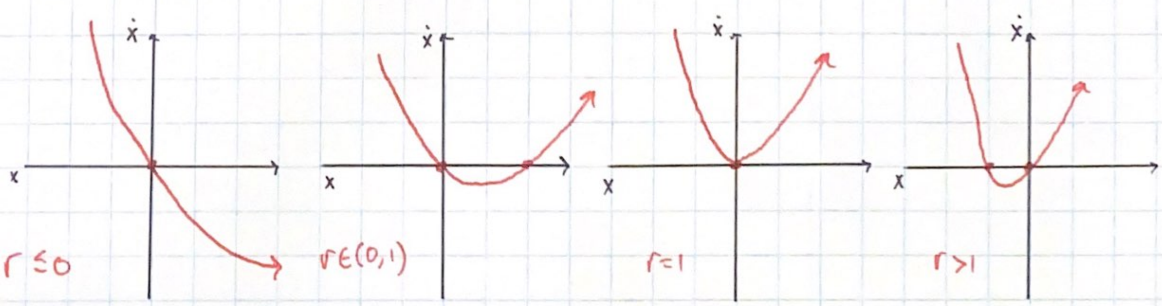


The Bifurcation diagram is then,



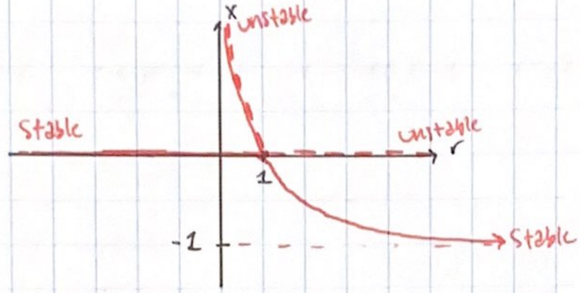
which consists of two saddle-node bifurcations at $r = \pm 2$.

c) $\dot{x} = rx - \log(1+x)$. First notice that when $x=0$, $\dot{x}=0$ which yields a permenent fixed point for any r. When $r<0$, we add two negative values when $x>0$, and positive values when $x<0$. When $r \in (0,1)$, we obtain a second fixed point. When $r=1$, $x - \log(1-x)$ is always non-negative, hence a semi-stable point at $x=0$. Lastly when $r>1$, a new fixed point is created and a transcritical bifurcation occurs.



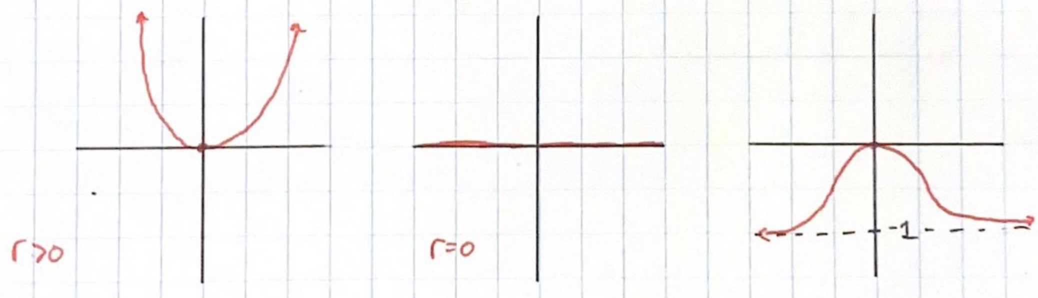
The corresponding vector fields are:

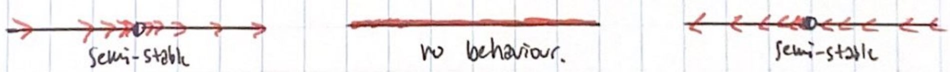
The bifurcation diagram is once again transcritical:



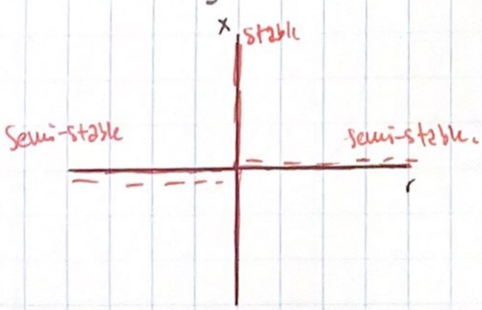
Stable/unstable fixed points swap.

3) $\dot{x} = e^{rx^2} - 1$. First notice that when $r=0$, $\dot{x}=0$ $\forall x$. This implies infinitely many fixed points. When $r>0$, we obtain symmetric exponential growth about $\dot{x}$ axis. When $r<0$, we have a Gaussian whose limits $x = \pm\infty$ arrive at $-1 = \dot{x}$. Then:





The corresponding bifurcation is transcritical, since the type of semi-stable behavior changes at $r=0$. This can also be thought of as a saddle-node bifurcation, since infinitely many fixed points are created when $r=0$.

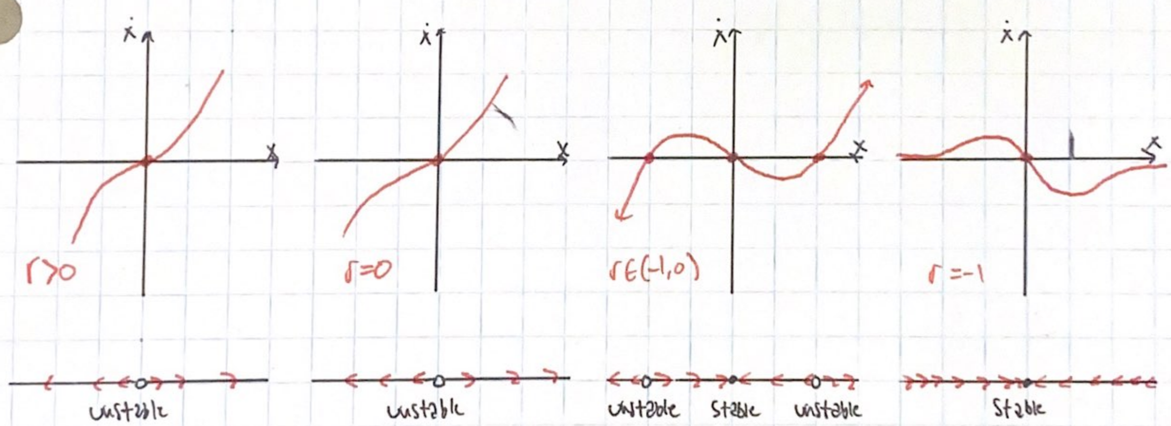


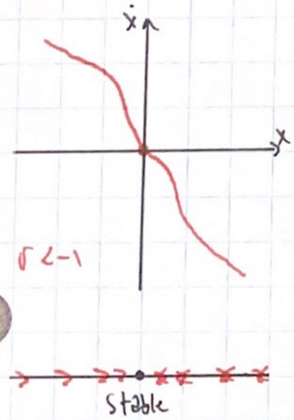
g) $\dot{x} = rx + \dfrac{x^3}{1+x^2}$. Multiplying the denominator out yields:

$$\dot{x} = \frac{rx(1+x^2) + x^3}{1+x^2} = \frac{x\left[r + x^2(1+r)\right]}{1+x^2}.$$

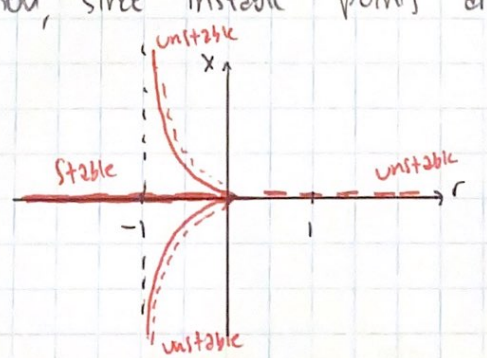
When $x=0$, $\dot{x}=0$ always thus $x=0$ is always a fixed point. The multiplicative term $r + x^2(1+r)$ yields that $x = \pm\sqrt{\frac{-r}{1+r}}$. Here, when $r \geq 0$, this term yields complex fixed points, so $x=0$ still remains the only fixed point.

When $r \in (-1,0)$, two more fixed points are created since $\pm\sqrt{\frac{-r}{1+r}} \in \mathbb{R}$. Lastly, when $r=-1$, or if $r<-1$, one critical point remains since $\sqrt{\frac{-r}{1+r}}$ is either undefined $(r=-1)$ or again complex $(r<-1)$.





Which corresponds to a subcritical pitchfork bifurcation, since instable points are created.



Q 3.7.6:

$$\dot{x} = -Kxy \quad , \quad \dot{y} = Kxy - ly \quad , \quad \dot{z} = ly .$$

a) To show that $x+y+z = N$, we must first consider that

$$\dot{x} + \dot{y} + \dot{z} = -Kxy + Kxy - ly + ly = 0,$$

and since the derivative of any constant is zero, integration with respect to t yields that

$$\int (\dot{x}+\dot{y}+\dot{z})\, dt = \int (0)\, dt \quad \rightarrow \quad x+y+z = N,$$

where $N \in \mathbb{R}$ is a constant.

Physically, N represents the total population count, including dead people, sick and healthy.

b) Now $\dot{x} = -Kxy$, but $y = \frac{\dot{z}}{l}$, so $\dot{x} = -Kx\frac{\dot{z}}{l}$.

Separation of the variables gives that

$$\frac{dx}{dt} \cdot \frac{1}{x} = -\frac{K}{l} \frac{dz}{dt}$$

or equivalently, that $\frac{dx}{x} = -\frac{K}{l} dz$. We may now turn to integration:

$$\int_{x_0}^{x} \frac{dx'}{x'} = \log|x| \Big|_{x_0}^{x} = -\frac{K}{l} z \implies \frac{x}{x_0} = \exp\left[-\frac{K}{l} z\right].$$

Finally, x_0 represents the initial number of healthy people, so we have that

$$x(t) = x_0 \exp\left(-\frac{K}{l} z(t)\right),$$

as required.

c) To show that $\dot{z} = l\left[N - z - x_0 \exp\left(-\frac{k}{l} z\right)\right]$, we must first consider that $\dot{z} = ly$, and $y = N - x - z$, so we find that

$$\dot{z} = l\left[N - x - z\right].$$

However since $x = x(t) = x_0 e^{-k/l \, z(t)}$, our equation for $\dot{z}$ becomes

$$\dot{z} = l\left[N - z - x_0 e^{-\frac{k}{l} z}\right],$$

which is what I wanted to show.

d) The dimensions of our problem only consist of n = a number of people, and t = time. Since $\dot{x} = -kxy$,

$$[\dot{x}] = \frac{n}{t} = -k \cdot n^2,$$

so k must have units $[k] = \frac{1}{nt}$.

Similarly, since $\dot{z} = ly$,

$$[\dot{z}] = \frac{n}{t} = l \cdot n,$$

so l must have units $[l] = \frac{1}{t}$.

This suggests that we introduce a dimensionless quantity

$$u = \frac{k}{l} z \quad , \quad [u] = \frac{[k]}{[l]}[z] = \frac{\frac{1}{nt}}{\frac{1}{t}} n = \text{dimensionless}.$$

Then $z = \frac{l}{k} u$ has dimensions of n. Then

$$\dot{z} = \frac{dz}{dt} = l\left[N - \frac{l}{k} u - x_0 e^{-u}\right].$$

Since N and $\frac{l}{k} u$ both have units of n, it suggests that we shall divide out of both sides the quantities λ and x_0:

$$\frac{dz}{dt} \cdot \frac{1}{\lambda x_0} = \frac{N}{x_0} - \frac{\lambda}{K x_0} u - e^{-u} .$$

Therefore the whole right hand side of this expression is dimensionless, and thus the left hand side shall be too. This motivates to introduce a dimensionless time τ.

Since $z = \frac{\lambda}{K} u$, then $dz = \frac{\lambda}{K} du$, so we first have that

$$\frac{dz}{dt} \cdot \frac{1}{\lambda x_0} = \frac{\lambda}{K} \frac{du}{dt} \cdot \frac{1}{\lambda x_0} = \frac{1}{K x_0} \frac{du}{dt} .$$

The expression $[K x_0 \, dt]^{-1}$ has units $[\frac{1}{nt} \cdot n \, dt]^{-1}$, which then implies we introduce $d\tau = K x_0 \, dt$, which is dimensionless. Therefore our initial expression has reduced to a dimensionless expression

$$\frac{du}{d\tau} = a - bu - e^{-u} ,$$

with $a = \frac{N}{x_0}$ and $b = \frac{\lambda}{K x_0}$, both dimensionless parameters, as required.

e) Since N represents the total number of people over the course of the epidemic (and N = constant), N = healthy + sick + dead, then N can <u>at most be x_0</u> at $t=0$, assuming every person is initially healthy. However, at a different initial time t_0', x_0'

may not represent the total number of (healthy) people. Therefore we will always find that $x_0 \leq N$, or that $1 \leq \frac{N}{x_0} = a$.

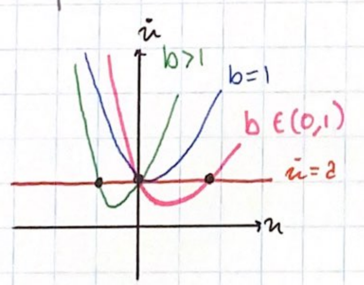
Similarly, since it is assumed that $k, l > 0$ are positive constants, and we assume $x_0 > 0$ (it wouldn't be an epidemic if $x_0 = 0$), then we will always have $\frac{l}{kx_0} > 0$ or $b > 0$ since x_0 cannot be negative either, which is what I wanted to show.

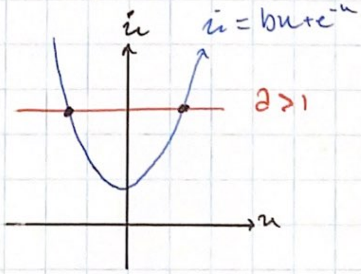
3). Again, we have that $\dot{u} = a - bu - e^{-u}$. We seek to determine the intersections of $a = bu + e^{-u}$. For $a = 1$, $b = 1$, the only solution $1 = u + e^{-u}$ is when $u = 0$. This is a semi-stable fixed point and $\dot{u}$ is negative outside of $u = 0$.

When $b \in (0,1)$, $1 = bu + e^{-u}$ has two solutions, at $u = 0$ and another since $bu + e^{-u} < 1$ in between this fixed points.

When $b > 1$, we again have two fixed points however this time on the negative axis. When $a > 1$, we will always have two fixed points.

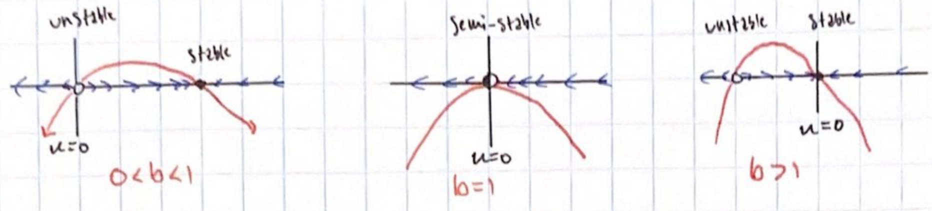
$a = 1$



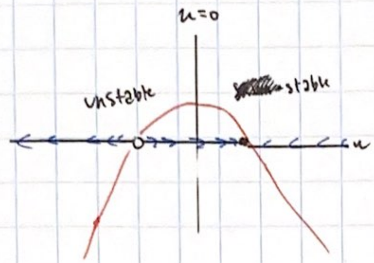


The corresponding vector fields are

$\underline{a=1}$



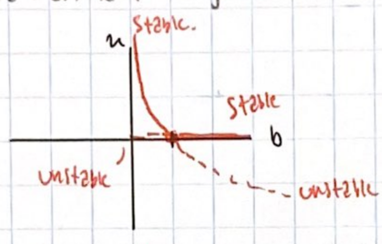
$\underline{a>1}$:



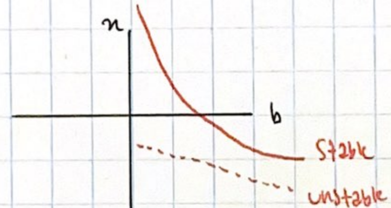
$\dot{x} = a - bx - e^{-x}$.

which yields a transcritical bifurcation when $a=1$:

$a=1$



$a>1$



g) Given $\dot{x} = a - bx - e^{-x}$, let $f(x) = a - bx - e^{-x}$. The maximum of $\dot{x}$ is given when $\frac{df}{dx} = 0$.

Now $f'(x) = -b + e^{-x}$, so $b = e^{-x}$ gives the maximum, or whenever $x = -\log(b)$.

Let this time of maximum be t_{peak}, so $x(t_{peak}) = -\log(b)$.

Allow me to recall our substitution $x = \frac{K}{2} z$, so $\dot{x} = \frac{K}{2}\dot{z}$.

This means that $\dot{x} \propto \dot{z}$. Similarly, since $\dot{z} = ly$, from our initial equations, then $\dot{x} \propto \dot{z}$ and $\dot{x} \propto y$, so we have that when $\dot{x}$ reaches its maximum at $t = t_{peak}$, so does $\dot{z}$ and y.

h) Again we have that $\dot{u}'(u) = -b + e^{-u}$.

When $b < 1$, then $\dot{u}'(u)$ is positive only on the interval $[0, -\log(b)]$, thus $\dot{u}'(u) = \ddot{u}$ is increasing on that interval. $\dot{u}'(u)$ decreases after, on the interval $(-\log b, \infty)$.

Assuming no initial deaths, $u(t=0) = 0$, so $\dot{u} = \frac{N}{x_0} \geq 1$ so $\dot{u}$ is increasing as before. When u hits its maximum, $u = -\log(b) \equiv u(t_{peak})$, so u attains a maximum in finite time.

In the limit as $t \to \infty$, $u(t) \to u_f(t_f)$, where $u_f(t_f)$ is the function satisfying $0 = a - bu_f - e^{-u_f}$. As seen in part (f), The function $\dot{u}$ attains a stable point as $t \to \infty$.

(i) When $b > 1$, the function determining the maximum $-\log(b)$ becomes negative, which is outside of our domain (the number of people can only be positive). Therefore the maximum occurs at $t=0$ since $a - bu - e^{-u}$ is monotonically decreasing as $u \to \infty$. Therefore $t_{peak} = 0$.

(j) When $b=1$, an epidemic occurs only if $a > 1$: that is, $\frac{N}{x_0} > 1$ or that there exists one initial sick person at minimum for $x_0 < x+y+z \to x_0 + y_0 + z_0$ at $t=0$. This is a highly unstable fixed point.

k) Our initial system is again

$$\dot{x} = -kxy$$

$$\dot{y} = kxy - ly$$

$$\dot{z} = ly.$$

The difference during covid-19 was that sick people can recover and be vaccinated. Then we revise the model:

$$\dot{x} = -kxy + myz$$

$$\dot{y} = kxy - ly - myz$$

$$\dot{z} = ly$$

where m represents the rate at which people recover from being sick, which is proportional to the number of people who have died (and hence cannot continue to transmit the illness) and the sick people (who recover if they are responsible and self-isolate).