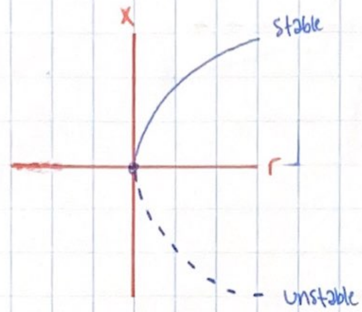
PHY460 Problem Set 11 Due: October 21, 2022

1) <u>3.4.5</u> $\dot{x} = r - 3x^2$.

This equation represents a <u>saddle-node bifurcation</u>. When $r<0$, there are no real roots. When $r=0$, the only (semi-stable) fixed point which is created is at $x=0$. When $r>0$, the two fixed points are

$$x = \pm\sqrt{\frac{r}{3}} \, .$$



The corresponding bifurcation diagram is depicted above.

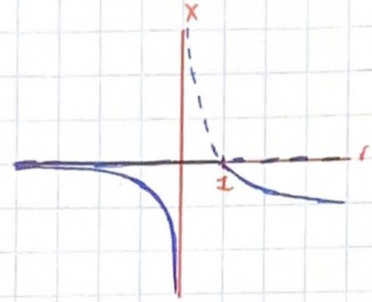
<u>3.4.6</u> $\dot{x} = rx - \frac{x}{1+x}$.

Factoring this expression yields $\dot{x} = x\left(r - \frac{1}{1+x}\right)$, thus fixed points are given by $0 = x\left(r - \frac{1}{1+x}\right)$. $x=0$ is hence always a fixed point, and $r - \frac{1}{1+x} = 0 \Rightarrow x = \frac{1}{r} - 1$ gives the location of the other fixed point.

When $r<0$, both fixed points are stable, yet when $r=0$, only the one stable fixed point at $x=0$ remains.

When $0<r<1$, an unstable fixed point is created on the positive side of the axis, until $r=1$ establishes a transcritical bifurcation when $r>0$.

The following plots and bifurcation diagram are shown:

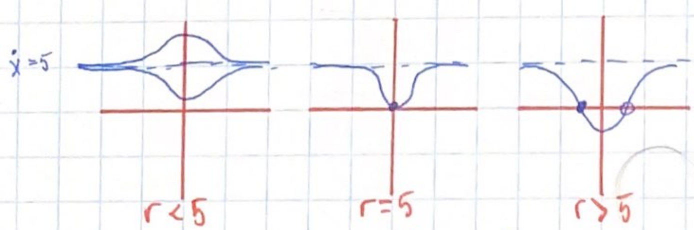


<u>Transcritical Bifurcation,</u> since the stability of fixed points "swap" as r is varied.

<u>3.4.7</u> $\dot{x} = 5 - re^{-x^2}$.

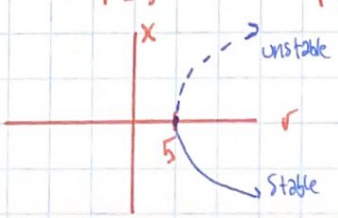
Notice that $5 - re^{-x^2}$ is an equation representing a Gaussian function whose asymptote is at $\dot{x} = 5$, which is seen by taking the limit as $x \to \pm\infty$. Here, r is the parameter which establishes the amplitude of the Gaussian at $x=0$.

Notice that when $r < 5$, no fixed points are created, that is, until $r=5$. Then $\dot{x} = 5 - 5e^{-x^2} = 0$ only when $x=0$. Whenever $r \geq 5$, two fixed points are created:



The corresponding bifurcation diagram depicts a <u>saddle-node bifurcation.</u>

Saddle-node Bifurcation.



3.4.8 $\quad \dot{x} = rx - \dfrac{x}{1+x^2}$.

Following a similar procedure as that of (3.4.6), I will proceed by factoring:

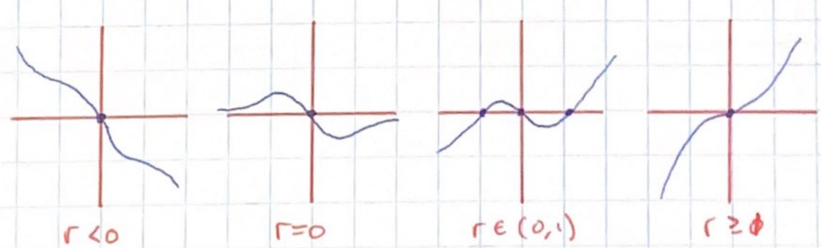
$$\dot{x} = 0 = x\left(r - \frac{1}{1+x^2}\right).$$

This implies that $x=0$ is always a fixed point, and so is $x = \pm\sqrt{\frac{1}{r} - 1}$, since $0 = r - \frac{1}{1+x^2}$ gives the other two fixed points.

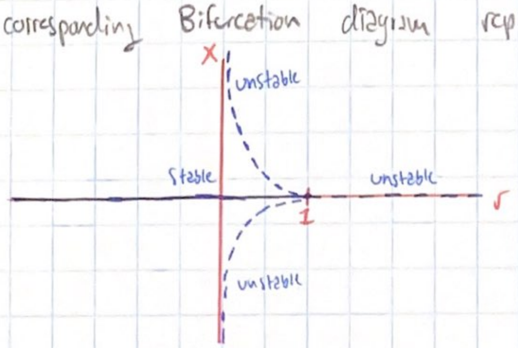
Note that when $r = 1$, the only fixed point is $x=0$.

When $\frac{1}{r} - 1 < 0$ or, when $r > 1$, the two fixed points no longer exist and the roots become complex. When $r < 0$, the equivalent occurs. $x = \pm\sqrt{\frac{1}{r} - 1}$ is undefined for $r = 0$, thus the two symmetrical fixed points are only defined for $r \in (0,1)$.

The slope of the function is generally positive when $r > 0$, zero when $r = 0$, and negative when $r < 0$. Plots are drawn below:



The corresponding Bifurcation diagram represents a <u>subcritical pitchfork bifurcation.</u>

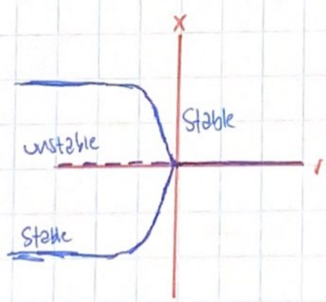


3.4.9 $\quad \dot{x} = x + \tanh(rx)$.

In terms of exponentials, $\dot{x}$ can be rewritten for an easier analyzation:
$$\dot{x} = x + \frac{e^{rx} - e^{-rx}}{e^{rx} + e^{-rx}} \equiv x + \frac{e^{2rx} - 1}{e^{2rx} + 1}.$$

Note that $\dot{x} = 0 \iff x = -\frac{e^{2rx} - 1}{e^{2rx} + 1}$. If $r \geq 0$, the only x (fixed point) satisfying this relation is $x = 0$. However, when $r < 0$, we can see that $\tanh(-x) = -\tanh(x)$, so there must be two other fixed points created by symmetry, since $-\tanh(x)$ conflicts with x (x is odd with positive slope, $-\tanh(x)$ is odd but with negative slope). The corresponding plots are:

And the bifurcation diagram depicts a <u>supercritical pitchfork bifurcation.</u>



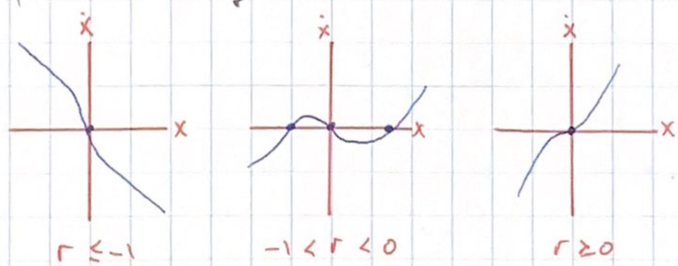
3.4.10 $\quad \dot{x} = rx + \frac{x^3}{1+x^2}$.

Applying a similar procedure once again (as used in 3.4.6, 3.4.8) I will proceed by factoring to find the fixed points.

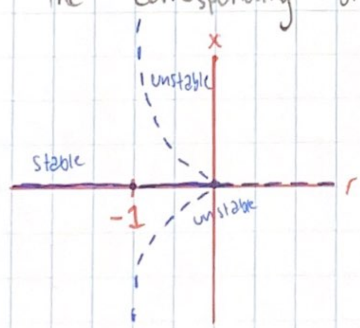
We have that $\dot{x} = rx + \frac{x^3}{1+x^2} = x\left(r + \frac{x^2}{1+x^2}\right)$. Now $\dot{x} = 0$ only when $x=0$ or $r = -\frac{x^2}{1+x^2}$ which implies that $x=0$ and $x = \pm\sqrt{\frac{-r}{r+1}}$ yield the fixed points.

When $-\frac{r}{r+1} \leq 0$, only one fixed point remains (at $x=0$). This correlates to $r \geq 0$. If $r+1 \leq 0$, a similar phenomenon occurs in which $x = \pm\sqrt{\frac{-r}{r+1}} \in \mathbb{C}$. Thus the only other fixed points (other than $x=0$) are created when $r \in (-1,0)$.

The plots are very similar to those of (3.4.8),



and the corresponding bifurcation diagram relates to a <u>subcritical pitchfork bifurcation.</u>



Q5) (i) Let $A = \begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix}$.

Compute A^n for any $n \in \mathbb{Z}$.

I will begin by general matrix multiplication:

$$A^2 = \begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & 0 \\ 2\lambda c & \lambda^2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} \lambda^2 & 0 \\ 2\lambda c & \lambda^2 \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^3 & 0 \\ 3\lambda^2 c & \lambda^3 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} \lambda^3 & 0 \\ 3c\lambda^2 & \lambda^3 \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^4 & 0 \\ 4c\lambda^3 & \lambda^4 \end{pmatrix}.$$

A general pattern is depicted as

$$\boxed{A^n = \begin{pmatrix} \lambda^n & 0 \\ nc\lambda^{n-1} & \lambda^n \end{pmatrix}.}$$

Now, $e^{At} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{pmatrix} \lambda^n & 0 \\ nc\lambda^{n-1} & \lambda^n \end{pmatrix}$.

Notice that the diagonals become $\sum_{n=0}^{\infty} \frac{t^n}{n!} \lambda^n = e^{\lambda t}$

and the off-diagonal becomes $\sum_{n=0}^{\infty} \frac{t^n}{n(n-1)!} nc\lambda^{n-1} = \sum_{n=0}^{\infty} (ct) \frac{t^{n-1}}{(n-1)!} \lambda^{n-1}$

$= cte^{\lambda t}$ and 0, respectively.

Therefore: $e^{At} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = \boxed{\begin{pmatrix} e^{\lambda t} & 0 \\ cte^{\lambda t} & e^{\lambda t} \end{pmatrix} \equiv \begin{pmatrix} C & D \\ E & G \end{pmatrix}}$

where $C = G = e^{\lambda t}$, $D = 0$, $E = cte^{\lambda t}$.

Since the time derivative $\frac{d}{dt}$ of a matrix is just the time derivative of each of the matrix's entries, then

$$\frac{d}{dt} e^{At} = \frac{d}{dt}\begin{pmatrix} e^{\lambda t} & 0 \\ cte^{\lambda t} & e^{\lambda t} \end{pmatrix} = \begin{pmatrix} \lambda e^{\lambda t} & 0 \\ ce^{\lambda t} + ct\lambda e^{\lambda t} & \lambda e^{\lambda t} \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix}\begin{pmatrix} e^{\lambda t} & 0 \\ cte^{\lambda t} & e^{\lambda t} \end{pmatrix}$$

which is equivalently $= Ae^{At}$, therefore

$\boxed{\frac{d}{dt} e^{At} = Ae^{At}}$, as required.

(ii) Now, we use the result that $e^{At} = \begin{pmatrix} e^{\lambda t} & 0 \\ cte^{\lambda t} & e^{\lambda t} \end{pmatrix}$ to find the solution of the equation

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{At} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}.$$

By simple matrix multiplication, we find that

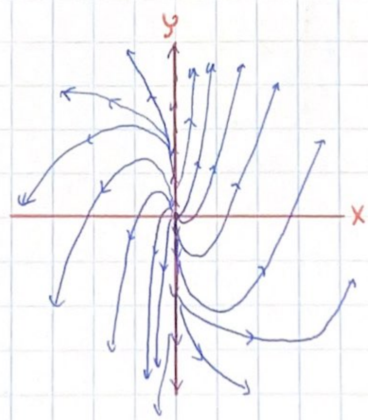
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{\lambda t} \\ ct\,e^{\lambda t} \end{pmatrix} x(0) + \begin{pmatrix} 0 \\ e^{\lambda t} \end{pmatrix} y(0),$$

or equivalently

$$\boxed{x(t) = e^{\lambda t}\cdot x(0) \quad , \quad y(t) = cte^{\lambda t}\cdot x(0) + e^{\lambda t}\cdot y(0).}$$

[$\lambda=2$, $c=3$] If $x(0)=0$, the solution diverges away from $(0,0)$ exponentially staying vertically with $x(t)=0$, $y(t) = e^{\lambda t}\cdot y(0)$.

If $y(0)=0$, the solution exponentially spirals outward for $x(0)\neq 0$.



Vector field plot of system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

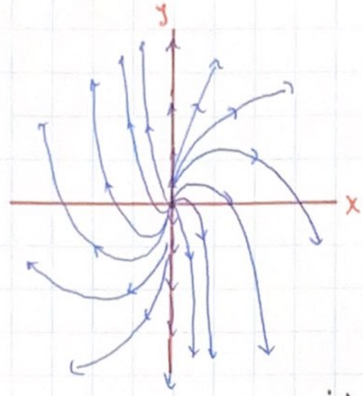
$\Delta = 4$, $\tau = 4$ so the fixed point is a $\boxed{\text{degenerate node.}}$

[$\lambda=2$, $c=-3$] Equivalent to the last system, $c \to -c$ just changes the direction of the flow of $y(t)$ as time proceeds.

The vector field plot of the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

correlates to



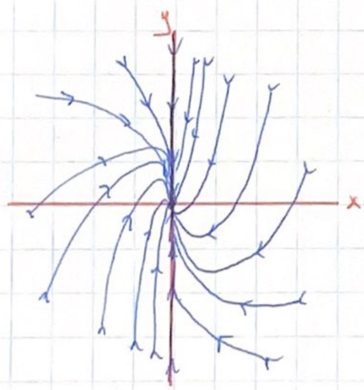
$\Delta = 4$, $\tau = 4$ thus

$\tau^2 - 4\Delta = 0$ corresponds to another $\boxed{\text{degenerate node.}}$

$[\lambda = -2,\ c = 2]$ In this system $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, the flow from the equations of motion trajectory

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{-2t} & 0 \\ 2te^{-2t} & e^{-2t} \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$$

corresponds to an inward flow to a stable point rather then an outward flow from an unstable fixed point.



$\Delta = 4$, $\tau = -4$

$\tau^2 - \Delta 4 = 16 - 16 = 0$

thus this fixed point is another degenerate node.

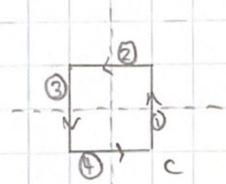
(iii) Index of the vector field around a circle. For all cases above $[\lambda=2,\ c=3]$, $[\lambda = 2,\ c=-3]$ and $[\lambda=-2,\ c=2]$ all present instances of vectors making a full rotation over a simple closed curve containing the origin. Thus I predict for each case above,

$\boxed{I_c = +1}$.

(iv) I will now proceed by calculating the index of the vector field around the origin for arbitrary values of c and λ, then solve numerically for each of the three cases described above.

Now, a circle may be a quite complicated closed curve to integrate a cartesian-defined vector field around, so instead I will use a square of side length 2 centred at the origin. The index of the vector field is given by the integral

$$I_C = \frac{1}{2\pi} \oint_C \frac{f\,dg - g\,df}{f^2 + g^2} .$$



Where the function f and g are given by

$$\dot{x} = f(x,y) = \lambda x \quad \text{and} \quad \dot{y} = g(x,y) = cx + \lambda y,$$

determined by the matrix equation $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

I will proceed by integrating each line segment. I will fully show my integration procedure for lines ① and ②, since ③ and ④ follow via the exact same procedure.

Now:

$$f(x,y) = \lambda x \qquad g(x,y) = cx + \lambda y$$
$$df = \lambda\, dx \qquad dg = c\,dx + \lambda\, dy .$$

Then,

① $x=1$ $\quad f=\lambda$ $\quad -1\le y\le 1$ $\quad g=c+\lambda y$ $\quad y: -1\to 1$

$dx=0$ $\quad df=0$ $\quad dy$ varies $\quad dg=\lambda\, dy$

$$\int_{-1}^{1}\frac{\lambda(\lambda dy)-(c+\lambda y)(0)}{\lambda^2+(c+\lambda y)^2}=\int_{-1}^{1}\frac{\lambda^2\, dy}{\lambda^2+(c+\lambda y)^2}$$

$$=\int_{-1}^{1}\frac{\lambda^2}{\lambda^2}\,\frac{dy}{1+(y+\frac{c}{\lambda})^2}=\arctan\left(y+\frac{c}{\lambda}\right)\Big|_{-1}^{1}.$$

② $-1\le x\le 1$ $\quad f=\lambda x$ $\quad y=1$ $\quad g=cx+\lambda$ $\quad x: 1\to -1$

dx varies $\quad df=\lambda dx$ $\quad dy=0$ $\quad dg=c\,dx$

$$\int_{1}^{-1}\frac{\lambda c x\, dx-(cx+\lambda)(\lambda\, dx)}{\lambda^2x^2+(cx+\lambda)^2}=-\int_{-1}^{1}\frac{-\lambda^2\, dx}{\lambda^2x^2+(cx+\lambda)^2}$$

$$=\int_{-1}^{1}\frac{\lambda^2\, dx}{\lambda^2x^2+c^2x^2+\lambda^2+2c\lambda x}.$$

Complete the square in the denominator:

$$(\lambda^2+c^2)x^2+2c\lambda x+\lambda^2 \longrightarrow \text{LET } p^2=\lambda^2+c^2$$

$$=p^2x^2+2c\lambda x+\left(\frac{c\lambda}{p}\right)^2-\left(\frac{c\lambda}{p}\right)^2+\lambda^2$$

$$=\left(px+\frac{c\lambda}{p}\right)^2+\lambda^2\left(1-\frac{c^2}{p^2}\right).$$

To integrate easier, let $u=px+\frac{c\lambda}{p}$; $du=p\,dx$ and let $K^2=\lambda^2\left(1-\frac{c^2}{p^2}\right)$.

$$=\int_{u(-1)}^{u(1)}\frac{\lambda^2\cdot\frac{1}{p}\,du}{u^2+K^2}=\frac{\lambda^2}{p}\int_{u(-1)}^{u(1)}\frac{du}{K^2\left(\frac{u^2}{K^2}+1\right)} \qquad \text{LET } v=\frac{u}{K},\ dv=\frac{1}{K}du$$

$$=\int_{v(u(-1))}^{v(u(1))}\frac{\lambda^2}{pK^2}\cdot K\,\frac{dv}{v^2+1}=\frac{\lambda^2}{pK}\arctan(v)\Big|_{v(u(-1))}^{v(u(1))}$$

→ Undo substitutions: $=\dfrac{\lambda^2}{p\left(\lambda^2/p^2\,(p^2-c^2)\right)^{1/2}}\arctan\left(\dfrac{px+\frac{c\lambda}{p}}{\left(\lambda^2/p^2\,(p^2-c^2)\right)^{1/2}}\right)\Big|_{-1}^{1}$

$$=\frac{\lambda^2}{\lambda(\lambda^2+c^2-c^2)^{1/2}}\arctan\left(\frac{p^2x+c\lambda}{\lambda(\lambda^2+c^2-c^2)^{1/2}}\right)\Big|_{-1}^{1}$$

$$=1\cdot\arctan\left(\frac{(\lambda^2+c^2)x+c\lambda}{\lambda^2}\right)\Big|_{-1}^{1}.$$

Integrals ③ and ④ follow similarly:

③ $x=-1$ $f=-\lambda$ $-1 \le y \le 1$ $g=-c+\lambda y$ $y: 1 \to -1$

$dx=0$ $df=0$ dy varies $dg=\lambda\, dy$

$$\int_1^{-1} \frac{-\lambda^2\, dy}{\lambda^2+(-c+\lambda y)^2} = \arctan\left(y-\frac{c}{\lambda}\right)\Big|_{-1}^{1}.$$

④ $-1 \le x \le 1$ $f=\lambda x$ $y=-1$ $g=cx-\lambda$ $x: -1 \to 1$

dx varies $df=\lambda dx$ $dy=0$ $dg=c\,dx$

$$\int_{-1}^{1} \frac{\lambda^2\, dx}{\lambda^2x^2+(cx-\lambda)^2} = \arctan\left(\frac{(\lambda^2+c^2)x^2-c\lambda}{\lambda^2}\right)\Big|_{-1}^{1}.$$

Now, we may evaluate each integral and add:

$$I_c = \frac{1}{2\pi}\left[I_1+I_2+I_3+I_4\right]$$

$$= \frac{1}{2\pi}\left[\arctan\left(1+\frac{c}{\lambda}\right)-\arctan\left(-1+\frac{c}{\lambda}\right)+\arctan\left(1-\frac{c}{\lambda}\right)-\arctan\left(-1-\frac{c}{\lambda}\right)\right.$$
$$\left.+\arctan\left(\frac{\lambda^2+c^2+c\lambda}{\lambda^2}\right)-\arctan\left(\frac{-\lambda^2-c^2+c\lambda}{\lambda^2}\right)+\arctan\left(\frac{\lambda^2+c^2-c\lambda}{\lambda^2}\right)-\arctan\left(\frac{-\lambda^2-c^2-c\lambda}{\lambda^2}\right)\right]$$

However, since $\arctan(-x) = -\arctan(x)$, factoring out a '$-1$' in the 2^{nd}, 4^{th}, 6^{th}, and 8^{th} terms tells us that

$$\boxed{I_c = \frac{1}{2\pi}\left[2\arctan\left(1+\frac{c}{\lambda}\right)+2\arctan\left(1-\frac{c}{\lambda}\right)+2\arctan\left(\frac{\lambda^2+c^2+c\lambda}{\lambda^2}\right)+2\arctan\left(\frac{\lambda^2+c^2-c\lambda}{\lambda^2}\right)\right]}$$

This can now be solved numerically.

$[\lambda=2,\ c=3]$ $I_c = \frac{1}{\pi}\left[\arctan\frac{5}{2}-\arctan\frac{1}{2}+\arctan\frac{19}{4}+\arctan\frac{7}{4}\right]$

$= \frac{\pi}{\pi} = 1.$

$[\lambda=2,\ c=-3]$ $I_c = \frac{1}{\pi}\left[-\arctan\frac{1}{2}+\arctan\frac{5}{2}+\arctan\frac{7}{4}+\arctan\frac{19}{4}\right]$

$= \frac{\pi}{\pi} = 1.$

$[\lambda=-2,\ c=2]$ $I_c = \frac{1}{\pi}\left[\arctan(2)+\arctan(0)+\arctan(1)+\arctan(3)\right]$

$= \frac{\pi}{\pi} = 1.$

This is as predicted, thus $(0,0)$ is a source or sink fixed point.

(v) In the case where nonlinear perturbations are added to the system, the behavior around the fixed point can most certainly change. Allow me to construct an example to illustrate various behaviors.

Consider the non-linear system defined by

$$\dot{x} = f(x,y) = \lambda x - \sin(cx)$$
$$\dot{y} = g(x,y) = cx + \lambda y,$$

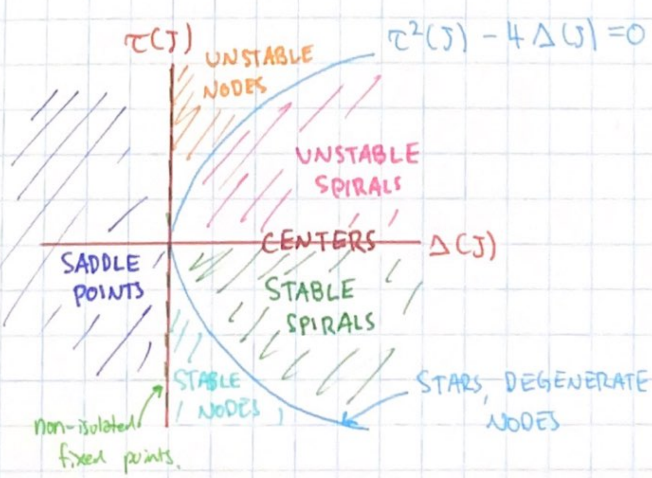
The added nonlinear term appearing as '$-\sin(cx)$'.

If this term were not added, we would indeed arrive at our initial system. Even still, $f(0,0) = g(0,0) = 0$ so $(0,0)$ is still a fixed point of the system. Linearizing around the origin tells us the behavior of the fixed point:

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} \lambda - c\cos(cx) & 0 \\ c & \lambda \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} \lambda - c & 0 \\ c & \lambda \end{pmatrix}.$$

Note that: $\Delta(J) = \lambda(\lambda - c)$ and $\tau(J) = 2\lambda + c$.

Allow me to now return to the fixed point classification diagram:



Now, since $\Delta(J) = \lambda(\lambda - c)$, if $\lambda - c < 0$ then $\Delta(J) < 0$, hence the fixed point converges to a saddle point.

If $2\lambda = -c$, then $\tau(J) = 0$ and $\Delta(J) = \lambda(\lambda + 2\lambda) = 3\lambda^2$ is always positive, hence we arrive at a 'centre' fixed point, or a stable node.

If $\tau^2(J) - 4\Delta(J) < 0$, spirals are obtained and stable/unstable nodes by $\tau^2(J) - 4\Delta(J) > 0$.

Each are depicted for different values of λ and c. If one were to add more freedom of parameters, you could define the nonlinear term to have an amplitude:

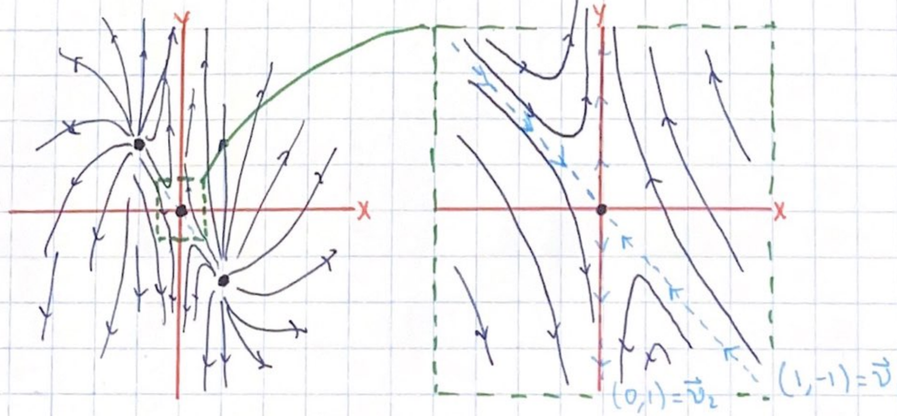
$$f(x,y) = \lambda x - A\sin(cx), \quad g(x,y) = cx + \lambda y.$$

For an example, take $\lambda = 2$, $c = 3$. Then

$$f(x,y) = 2x - \sin(3x), \quad g(x,y) = 3x + 2y.$$

The matrix linearization is then $\begin{pmatrix} -1 & 0 \\ 3 & 2 \end{pmatrix}$. Now, $\Delta = -2$ and $\tau = 1$, which corresponds to a saddle-point at $(0,0)$.

The vector field is plotted below:



Note the index change of $(0,0)$, from $+1 \to -1$ once the perturbation term has been added.

$\bullet$: fixed point of system.

Q6 | 5.2.4

$\dot{x} = 5x + 10y \qquad \dot{y} = -x - y.$

$\rightarrow M = \begin{pmatrix} 5 & 10 \\ -1 & -1 \end{pmatrix}$ so $\dot{\vec{x}} = M\vec{x}$.

Eigenvalues of M: $\det(M - \lambda I) = \begin{vmatrix} 5-\lambda & 10 \\ -1 & -1-\lambda \end{vmatrix}$

$= (5-\lambda)(-1-\lambda) + 10 = -5 - 5\lambda + \lambda + \lambda^2 + 10$

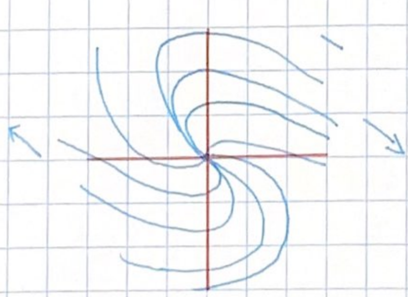
$= \lambda^2 - 4\lambda + 5 \longrightarrow \lambda = \dfrac{4 \pm \sqrt{16 - 4(1)(5)}}{2} = \dfrac{4 \pm 2i}{2} = 2 \pm i.$

Now, $\Delta(M) = -5 + 10 = 5$ and $\tau(M) = 4.$

This implies that $4^2 - 4(5) = 16 - 20 = -4$. Since

$\tau(M) = 2\sqrt{\Delta} = 2\sqrt{10} > 4$ and $\tau(M) = 4 > 0$, we obtain an

$\boxed{\text{unstable spiral}}$ at the origin, similar to that of a degenerate node:



Eigenvalues complex:

$\lambda = 2 \pm i$

Clockwise rotation: $\dot{x}(1,0) = 5 \quad \dot{y}(1,0) = -1 \quad \searrow$

$\dot{x}(-1,0) = -5 \quad \dot{y}(-1,0) = 1 \quad \nwarrow$

5.2.5

$\dot{x} = 3x - 4y, \quad \dot{y} = x - y \quad \rightarrow \quad M = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}.$

Eigenvalues of M: $|M - \lambda I| = \begin{vmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) + 4$

$= -3 - 3\lambda + \lambda + \lambda^2 + 4 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)(\lambda - 1). \quad \lambda = 1$, (multiplicity 2)

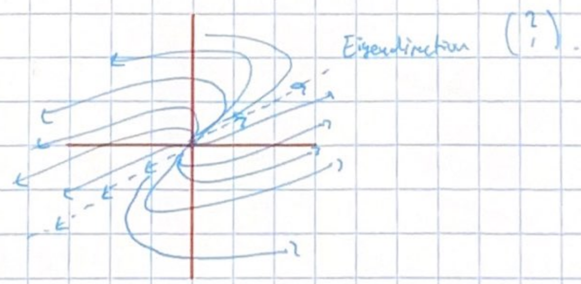
Here, we obtain an unstable point at $(0,0)$ since both eigenvalues are positive (moving away from 0). We now proceed by finding the eigenvector(s) of the system.

Finding the eigenvectors of the matrix then determine the direction of the flow:

① $\ker(M-I) = \ker\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} = \ker\begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}$ by row reduction. For

$$\begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0,$$ then $v_1 = 2v_2$ so the eigenvector is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

The determinant of M is $\Delta(M) = 1$ and the trace is $\tau(M) = 2$. Here, $\tau^2(M) - 4\Delta(M) = 4 - 4 = 0$ therefore this point lies along the star-degenerate curve, hence this fixed point is a $\boxed{\text{degenerate node}}$. $M\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$ ↙, $M\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and thus the 'spiral' is counter clockwise.

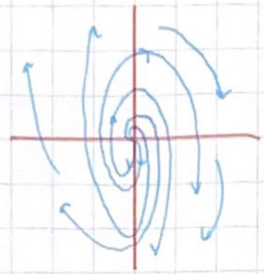


5.2.7

$\dot{x} = 5x + 2y$, $\dot{y} = -17x - 5y$. — $\dot{\vec{x}} = \begin{pmatrix} 5 & 2 \\ -17 & -5 \end{pmatrix}\vec{x}$, with

$M = \begin{pmatrix} 5 & 2 \\ -17 & 5 \end{pmatrix}$. Now $\Delta(M) = 25 + 34 = 59$. $\tau(M) = 10$, with this information, since $4\Delta(M) = 236 > \tau(M)^2 = 100$, then this point $(59, 10)$ lies in the unstable spiral region. Since $M\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -17 \end{pmatrix}$ and $M\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$, the vectors must be rotating clockwise.

The corresponding phase plot depicts an $\boxed{\text{unstable spiral}}$ at $(0,0)$.



The eigenvalues of M are given by $|M - \lambda I| = (5-\lambda)(-5-\lambda) + 34$

$= -25 + 34 + \lambda^2$ or $\lambda = \pm i\sqrt{9} = \pm 3i$. The eigenvectors are

$\ker|M + 3iI| = \ker\begin{pmatrix} 5+3i & 2 \\ -17 & -5+3i \end{pmatrix} \to \ker\begin{pmatrix} 5+3i & 2 \\ 0 & 0 \end{pmatrix}$

which corresponds to the eigenvectors $\begin{pmatrix} -5 \pm 3i \\ -17 \end{pmatrix}$. Since one eigenvector is much less than the other in terms of complex components, the spiral will be slightly 'stretched' in one direction.

5.2.10

$\dot{x} = y, \quad \dot{y} = -x - 2y \longrightarrow M = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$.

Here, $\Delta(M) = 1$ and $\tau(M) = -2$ which corresponds to a $\boxed{\text{degenerate node}}$ on the $\Delta - \tau$ plot, since $\Delta(M) = 1$. The eigenvalues of M are $\det(M - \lambda I) = -\lambda(-2-\lambda) + 1 = 2\lambda + \lambda^2 + 1 = (\lambda + 1)$ with multiplicity 2.

Now, $\ker(M - I) \Rightarrow \begin{pmatrix} +1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$ implies $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is the eigenvector.

At $M\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $M\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, so the node is a 'spiral' clockwise shape: