PHY460 Problem Set 3 — Due November 28, 17:00

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6.3.11

(a) For a system given in polar coordinates $\begin{cases} \dot{r} = -r \\ \dot{\theta} = \dfrac{1}{\log(r)} \end{cases}$, one can explicitly solve for $r(t)$ and $\theta(t)$.

First note that $\dot{r} = -r$, which by a seperating of variables yields $\dfrac{dr}{r} = dt$, which implies that $\log|r| - \log|r_0| = t_0 - t$. Raising each to the exponent base e gives that $r(t) = r_0 e^{-t}$.
One may apply a similar process for the equation of motion for $\theta(t)$.

$$\frac{d\theta}{dt} = \frac{1}{\log(r)} = \frac{1}{\log(r_0) + t_0 - t} \implies \int_{\theta_0}^{\theta} d\theta' = \int_{t_0}^{t} \frac{dt'}{\log(r_0) + t_0 - t'}.$$

Thus $\theta - \theta_0 = \log\left|\log(r_0) + t_0 - t\right| - \log\left|\log(r_0)\right|$, and simplification implies that $\theta(t) = \theta_0 + \log\left|1 + \dfrac{t_0 - t}{\log(r_0)}\right|$. Therefore

$$\boxed{r(t) = r_0 e^{-t} \qquad \text{and} \qquad \theta(t) = \theta_0 + \log\left|1 + \frac{t_0 - t}{\log(r_0)}\right|}$$

(b) It is easy to see that for $r(t) = r_0 e^{-t}$, that as $t \to \infty$ then $r(t) \to 0$. By a similar method, note that

$$\begin{aligned} \lim_{t\to\infty} |\theta(t)| &= \lim_{t\to\infty} \left|\theta_0 + \log\left|1 + \frac{t_0 - t}{\log(r_0)}\right|\right| \\ &= \infty, \end{aligned}$$

since $\lim\limits_{t\to\infty} \left|1 + \dfrac{t_0 - t}{\log(r_0)}\right| = \infty$, and $\lim\limits_{a\to\infty} \log|a| = \infty$, which is what I wanted to show.

(c) To write the system in (x, y) coordinates, one may simply apply the change of basis

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan\left(\frac{y}{x}\right).$$

We have that

$$\begin{aligned} \dot{r} &= \frac{d}{dt}[\sqrt{x^2 + y^2}] \\ &= \frac{(2x\dot{x} + 2y\dot{y})}{2\sqrt{x^2 + y^2}} \\ &= \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2 + y^2}} \end{aligned}$$

which must be equal to $-r$ from the equation of motion. We thus obtain our first relation

$$x\dot{x} + y\dot{y} = -(x^2 + y^2). \tag{1}$$

Similarly for θ,

$$\begin{aligned}
\dot{\theta} &= \frac{1}{y^2/x^2 + 1} \frac{x\dot{y} - y\dot{x}}{x^2} \\
&= \frac{x\dot{y} - y\dot{x}}{x^2 + y^2}.
\end{aligned}$$

Setting this equal to $\dfrac{1}{\log(\sqrt{x^2 + y^2})}$ by the equations of motion in the system, we obtain the second relation

$$x\dot{y} - y\dot{x} = \frac{2(x^2 + y^2)}{\log(x^2 + y^2)}. \tag{2}$$

I have obtained two equations with two unknowns which I want to solve for: $\dot{x}, \dot{y}$.
First, equation (1) gives that $\dot{x} = (-(x^2 + y^2) - y\dot{y})\dfrac{1}{x}$. Substituting this into (2) lets us solve for $\dot{y}$:

$$\begin{aligned}
x\dot{y} - \frac{y}{x}(-(x^2 + y^2) - y\dot{x}) = x\dot{y} + \frac{y}{x}(x^2 + y^2 + y\dot{y}) &= \frac{2(x^2 + y^2}{)}\log(x^2 + y^2) \\
\implies x^2\dot{y} + y(x^2 + y^2) + y^2\dot{x} = \dot{y}(x^2 + y^2) + y(x^2 + y^2) &= (\dot{y} + y)(x^2 + y^2) \\
\dot{y} + y = \frac{2}{\log(x^2 + y^2)} \implies \dot{y} &= \frac{2}{\log(x^2 + y^2)} - y. \qquad (3)
\end{aligned}$$

Doing the same for $\dot{x}$, with our solution for $\dot{y}$ now, we have that

$$\begin{aligned}
\dot{x} &= -\frac{1}{x}\left(x^2 + y^2 + y\left(\frac{2}{\log(x^2 + y^2)} - y\right)\right) \\
&= -\frac{1}{x}\left(x^2 + y^2 - y^2 + \frac{2y}{\log(x^2 + y^2)}\right) \\
&= \frac{1}{x}\left(x^2 + \frac{2y}{\log(x^2 + y^2)}\right) \implies \dot{x} = -x - \frac{2y}{x\log(x^2 + y^2)}, \qquad (4)
\end{aligned}$$

and thus (3) and (4) give the system in cartesian coordinates.

(d) To linearize the system, let us first compute the Jacobian matrix of the cartesian system. Let

$$\begin{aligned}
F(x, y) &= -x - \frac{2y}{x\log(x^2 + y^2)} \\
G(x, y) &= -y + \frac{2}{\log(x^2 + y^2)}.
\end{aligned}$$

It is easy to see that the nonlinear perturbations are determined by the second terms in each of the functions, however it may be more rigorous to show this explicitly. We have that

$$J = \begin{pmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\ \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial y} \end{pmatrix}\Bigg|_{(0,0)}$$

$$
= \left. \begin{pmatrix} \dfrac{2y}{x^2\log(x^2+y^2)} + \dfrac{4y}{(x^2+y^2)\log(x^2+y^2)} - 1 & \dfrac{4y^2}{x(x^2+y^2)(\log(x^2+y^2))^2} - \dfrac{2}{x\log(x^2+y^2)} \\ -\dfrac{4x}{(x^2+y^2)(\log(x^2+y^2))^2} & -\dfrac{4y}{(x^2+y^2)\log(x^2+y^2)} - 1 \end{pmatrix} \right|_{(0,0)}
$$

$$
= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},
$$

which is what I wanted to show. Note that I computed these derivatives via derivative calculator, and I do not wish to explicitly write the process. Therefore the origin is a stable star for the linearized system.

6.3.13

We begin with the system $\begin{cases} \dot{x} = -y - x^3 \\ \dot{y} = x \end{cases}$. Note that this system has a fixed point at the origin, and hence the Jacobian around the origin is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This matrix has determinant 1 and trace 0, which corresponds to a center on the stability classification diagram. To show that this system is truly a nonlinear stable spiral, one may find a Liapunov function in the neighbourhood of $(0,0)$ and show that $\dot{V} < 0$ around the fixed point, but I will instead use the method of transforming the system into polar coodinates and showing explicitly that $\dot{r}$ is negative and $\dot{\theta}$ is non-zero. Letting $x = r\cos\theta$ and $y = r\sin\theta$, we find that

$$\begin{aligned}
\dot{r}\cos\theta - r\sin\theta\dot{\theta} &= -r\sin\theta - r^3\cos^3\theta \\
\dot{r}\sin\theta + r\cos\theta\dot{\theta} &= r\cos\theta.
\end{aligned}$$

We have obtained two equations with two unknowns, and hence we may proceed by solving for $\dot{r}$ and $\dot{\theta}$. We have that

$$\begin{aligned}
\dot{r} &= \frac{-(r\sin\theta + r^3\cos^3\theta) + r\sin\theta\dot{\theta}}{\cos\theta} \\
\implies r\cos\theta &= \frac{\sin\theta}{\cos\theta}\left[r\sin\theta\dot{\theta} - (r\sin\theta + r^3\cos^3\theta)\right] + r\cos\theta\dot{\theta} \\
\implies r\cos^2\theta &= r\sin^2\theta\dot{\theta} - r\sin^2\theta - r^3\cos^3\theta\sin\theta + r\cos^2\theta\dot{\theta} \\
\implies r\dot{\theta} - r &= r^3\cos^3\theta\sin\theta + 1 \\
\implies \dot{\theta} &= r^2\cos^3\theta\sin\theta + 1, \qquad (1)
\end{aligned}$$

which is always time-dependent and non-zero for sufficiently small $r \geq 0$ in the neighbourhood of the origin.
Solving for $\dot{r}$, we have that

$$\begin{aligned}
\dot{r} &= \frac{1}{\cos\theta}\left[-r\sin\theta - r^3\cos^3\theta + r\sin\theta(r^2\cos^3\theta\sin\theta) + r\sin\theta\right] \\
&= -r^3\cos^2\theta + r^3\sin^2\theta\cos^2\theta \\
&= -r^3\cos^2\theta(1 - \sin^2\theta) \\
&= -r^3\cos^4\theta, \qquad (2)
\end{aligned}$$

which is always negative since $r \geq 0$ and $\cos^4\theta \geq 0$, and therefore the radial velocity is pointing towards the origin.
As desired, I have shown that the origin is actually a stable spiral instead of a center, which is what we found by linearization.

3. Driven Pendulum

For the following problem, I will consider the equations of motion $\ddot{x} + \alpha\dot{x} + \sin x = \Gamma$ and $\ddot{x} + \alpha|\dot{x}|\dot{x} + \sin x = \Gamma$, whose respective systems are given by

$$(1)\begin{cases}\dot{x} = y \\ \dot{y} = -\alpha y - \sin x + \Gamma\end{cases} \qquad \text{and} \qquad (2)\begin{cases}\dot{x} = y \\ \dot{y} = -\alpha|y|y - \sin x + \Gamma\end{cases}$$

(a) Consider the system (1) as determined above. Allow me to fix $\alpha = 0.3$ and begin with $\Gamma = 2$. By numerically computing the phase portrait, a decrease in Γ will result in the system's stable limit cycle being annihilated in a homoclinic bifurcation:

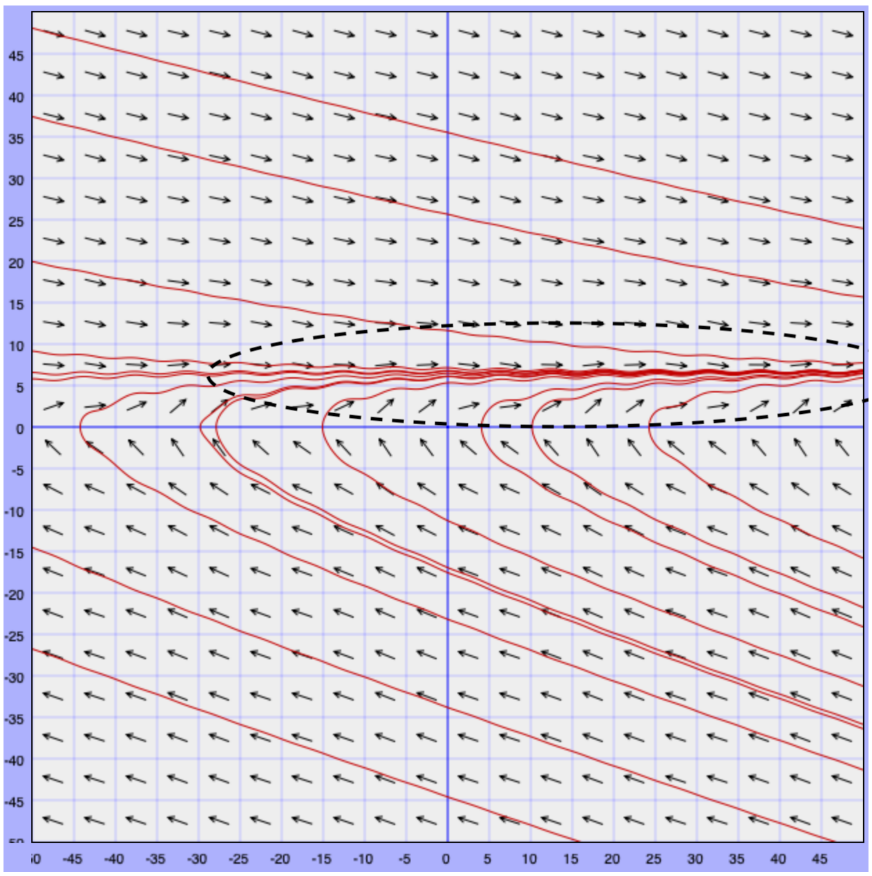


Figure 1: $\Gamma = 2$, every trajectory converges to limit cycle.

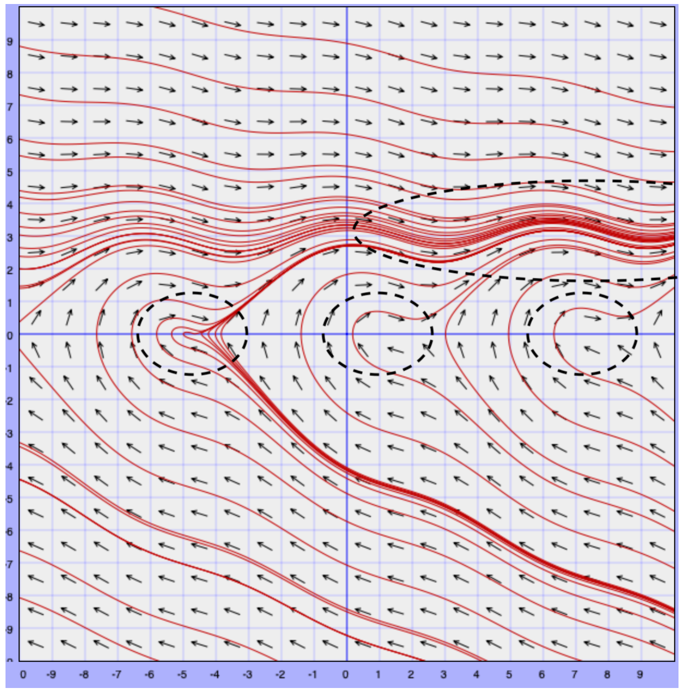


Figure 2: $\Gamma = 1$, every trajectory converges to limit cycle, yet trajectories which begin with $y_0 < 0$ slow around certain points along the $y = 0$ axis. A bifurcation is beginning to occur at these points.

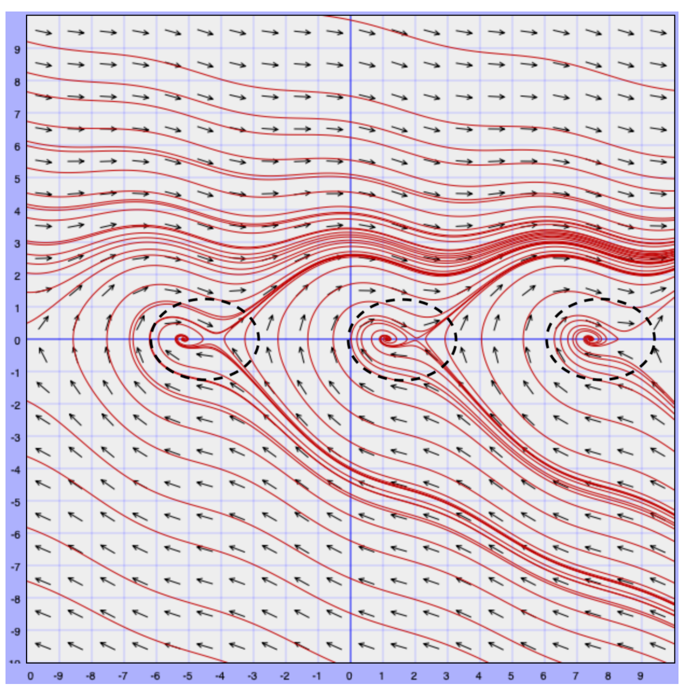


Figure 3: $\Gamma = 0.9$, various trajectories spiral into newly created fixed points. Note that two new fixed points have been created: one fixed, the other a saddle. Some trajectories converging to the limit cycle and some into the fixed points, depending on the initial condition.

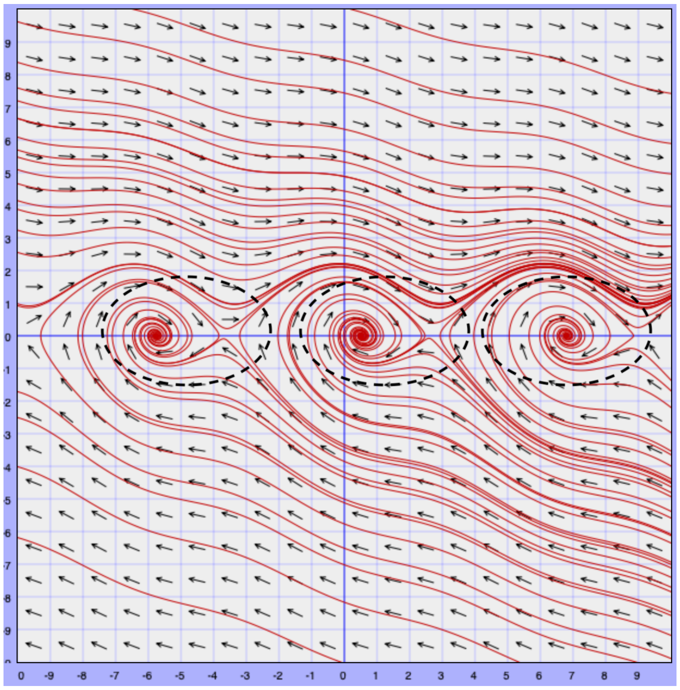


Figure 4: $\Gamma = 0.5$, various trajectories spiral into newly created fixed points, however this is more pronounced. The spirals and saddles are easier to see, and the behaviour as $t \to \infty$ is entirely dependent on the initial condition.

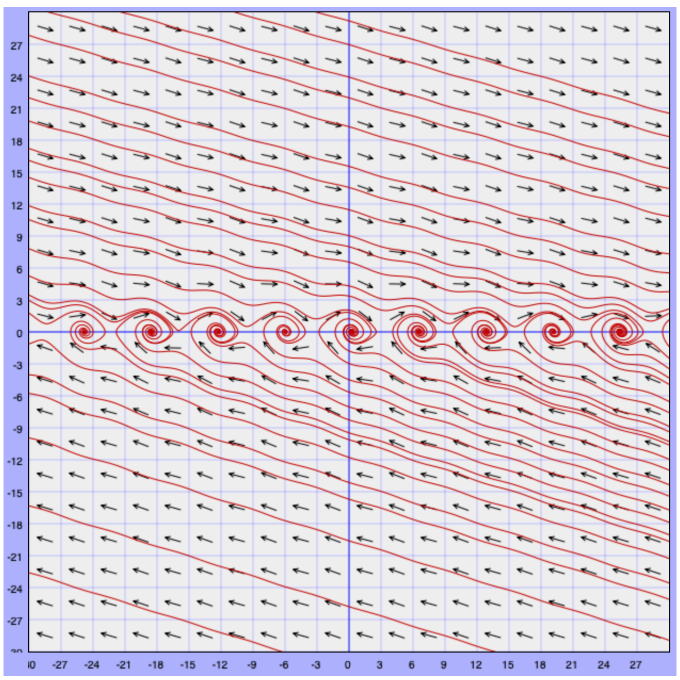


Figure 5: $\Gamma = 0.3$, every trajectory converges to a stable spiral, regardless the initial condition. A homoclinic bifurcation has occured, where saddle points have been created and trajectories saddle into either two of the stable states: the fixed point or the limit cycle. Here, the limit cycle has been destroyed throughout this process.

Allow me to now turn to the case where the damping term is very large, $\alpha \gg 1$. Taking $\alpha = 5$, one can see that fixed points and saddle points both occur on the trajectory of the limit cycle, hence resulting in an infinite period bifurcation:

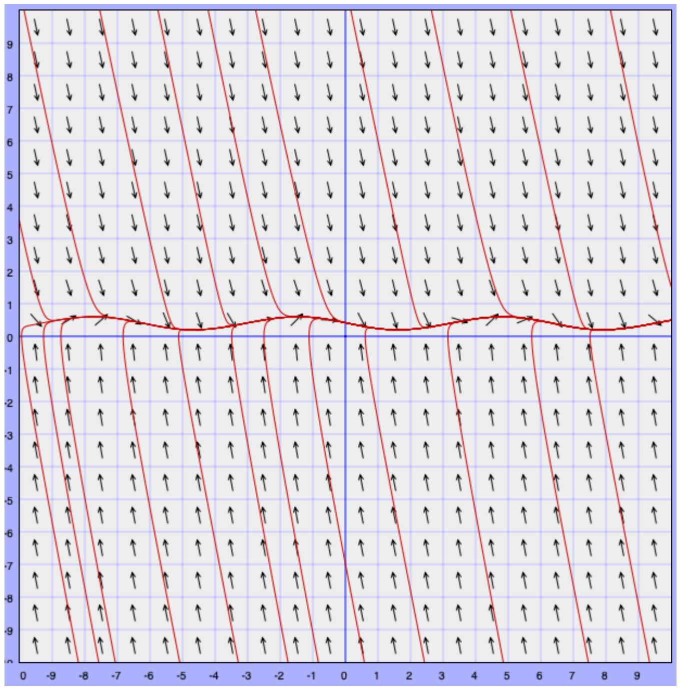


Figure 6: $\Gamma = 2, \alpha = 5$. Every trajectory converges to the limit cycle.

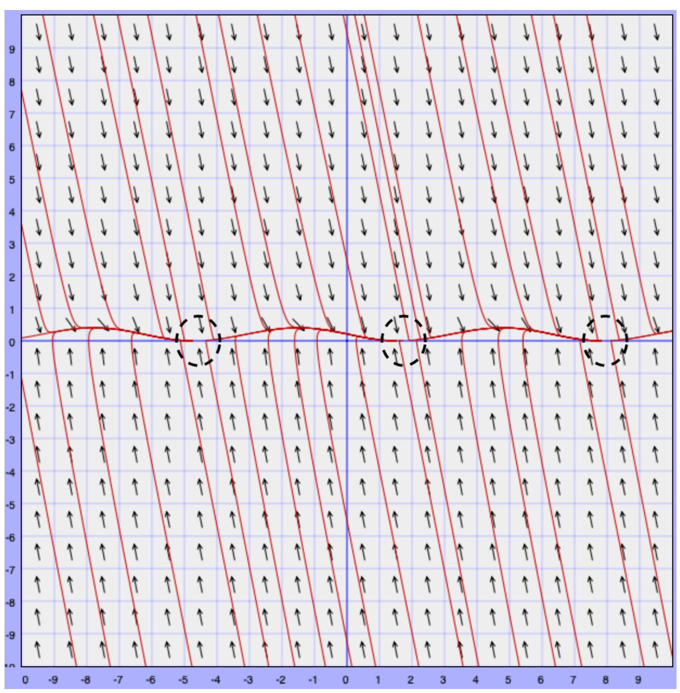


Figure 7: $\Gamma = 1, \alpha = 5$. Saddle points and fixed points begin appearing on the limit cycle trajectory, and every iniial condition results in convergence to a fixed point.

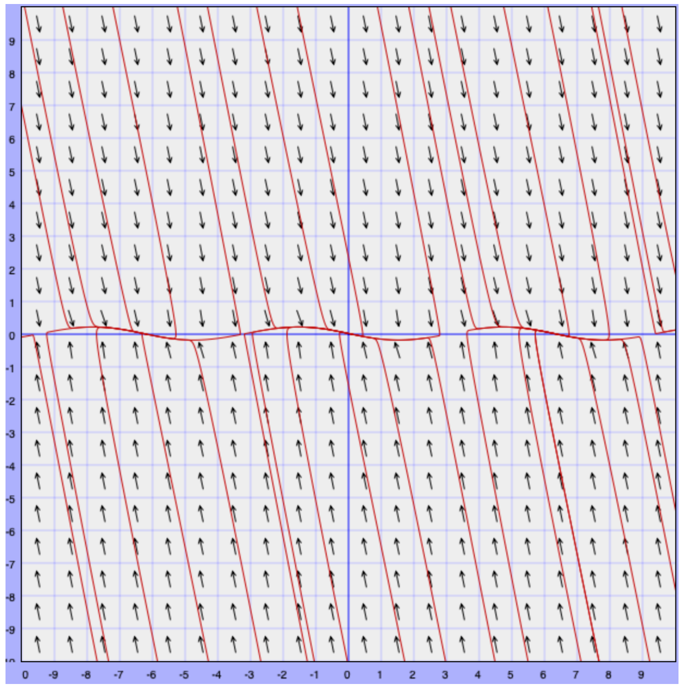


Figure 8: $\Gamma = 0.1, \alpha = 5$. Every trajectory converges to a fixed point, and the limit cycle is no destroyed in an inifite-period bifurcation.

In either case, the system's stable limit cycle is destroyed in a bifurcation as the constant torque Γ is decreased.

(b) Now let us turn to system (2). By definition, the fixed points of a system are given when both $(\dot{x}, \dot{y}) = (0, 0)$ and occur at (x^*, y^*). For (2), $\begin{cases} \dot{x} = y \\ \dot{y} = -\alpha|y|y - \sin x + \Gamma \end{cases}$, the fixed points of the system are always given by $y = 0$, else $\dot{x}$ would not be fixed. That is, all fixed points lie along the $x-$axis. Hence if $y = 0$, then $\dot{y} = 0 = -\sin x + \Gamma$ which implies that the fixed points are given whenever $y = 0$ and $\sin x = \Gamma$.
The two solutions to such an equation are found by taking the arcsin, that is,

$$x^* = \arcsin \Gamma, \pi - \arcsin \Gamma, \quad y^* = 0,$$

since two solutions must exist. These solutions repeat periodically in the x direction due to the periodicity of the sinusiodal functions, however I will only focus on the two fixed points since the others are the same in the phase plane. To classify these fixed points, one may take the Jacobian of the system and evaluate it at the fixed points (x^*, y^*). We have that

$$J = \begin{pmatrix} 0 & 1 \\ -\cos x & -\alpha|y| - \alpha y \begin{cases} 1 \ (y > 0) \\ -1 \ (y < 0) \end{cases} \end{pmatrix} \Bigg|_{(x^*, y^*)} = \begin{pmatrix} 0 & 1 \\ -\cos x & 0 \end{pmatrix} \Bigg|_{x = \arcsin \Gamma,\, x = \pi - \arcsin \Gamma}$$

The fixed point given by $(\pi - \arcsin \Gamma, 0)$ is the matrix

$$J_1 = \begin{pmatrix} 0 & 1 \\ -\cos(\pi - \arcsin(\Gamma)) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \sqrt{1 - \Gamma^2} & 0 \end{pmatrix}.$$

This Jacobian has $\mathrm{tr}(J_1) = 0$ and $\det(J_1) = -\sqrt{1+\Gamma^2}$, which corresponds to a saddle point in the fixed point classification diagram. Likewise, The fixed point given by $(\arcsin\Gamma, 0)$ is the matrix

$$J_2 = \begin{pmatrix} 0 & 1 \\ -\cos(\arcsin(\Gamma)) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\sqrt{1-\Gamma^2} & 0 \end{pmatrix}.$$

This Jacobian has $\mathrm{tr}(J_1) = 0$ and $\det(J_1) = \sqrt{1+\Gamma^2}$, which corresponds to a center! However, this is not what we had observed in the phase portrait, since we know a decrease in Γ creates a bifucation which gives rise to stable spiral points along the $y = 0$ axis. Therefore this must be a linearized center, so we must proceed by another method of showing the stability of this fixed point is actually stable and not a center.

Liapunov function. In the neighbourhood of the fixed point, one may approximate the stabilility by means of a Liapunov function. This is my process of finding a Liapunov function. I first considered a nonlinear pendulum with no damping or driven torque, whose equation of motion is given by $\ddot{x} + \sin x = 0$. The corresponding system is $\begin{cases} \dot{x} = y \\ \dot{y} = -\sin x \end{cases}$. The fixed points of the system are every $(k\pi, 0)$ for $k \in \mathbb{Z} \cup \{0\}$. For even k, we obtain stable linear centers, and odd k we obtain saddle points (I should not bother proving this; this is first year physics). We wish to find a positive-definite function V such that

- $V(x, y) > 0$ for all $(x, y) \neq (x^*, y^*)$ and $V(x^*, y^*) = 0$.
- $\dot{V}(x, y) < 0$ for all $(x, y) \neq (x^*, y^*)$.

This implies that the fixed point is indeed stable. For a nonlinear simple pendulum,

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = \frac{\partial V}{\partial x}(y) + \frac{\partial V}{\partial y}(-\sin x).$$

If we can somehow elimitate the $\sin x$ term and introduce a negative sign, we will obtain a zero $\dot{V}$, since each fixed point is a stable linear center. Setting $\frac{\partial V}{\partial x} = \sin x$ (for the result to be zero) and $\frac{\partial V}{\partial y} = y$, and integrating, we obtain that $V = \frac{y^2}{2} - \cos(x) + 2$ (to obtain positive-definiteness, one may add the constant 2 as I did). This is the Liapunov function for a simple nonlinear pendulum (see [13. Liapunov Functions]), and we can now apply this to our nonlinear driven system.

Starting with $V(x, y) = \frac{y^2}{2} - \cos x$, we must now account for the damping term in the equation of motion. If the fixed point stability is stable, then $\dot{V} < 0$ and not equal to zero as in the simple nonlinear pendulum case above. Note that

$$\begin{aligned} \dot{V} &= y\dot{y} + \sin x\dot{x} \\ &= y(-\alpha|y|y - \sin x + \Gamma) + \sin xy \\ &= -\alpha|y|y^2 + \Gamma y. \end{aligned}$$

This is not entirely negative, since y can be positive or negative in the neighbourhood of the fixed point. To fix this, I will try to annihilate this term by subtracting a $-\Gamma y$. The integral with respect to time of this term is $-\Gamma x$. Adding this term to our Liapunov function accounts for the nonlinear term in order for $\dot{V} < 0$:

$$V(x, y) = \frac{y^2}{2} - \cos x - \Gamma x + C$$

where C is a positive constant which accounts for the positive-definiteness of the Liapunov function. Adding any other terms with x or y terms may alter the $\dot{V}$ term, which may change it's sign. Hence we will stick with this general form of function.

To determine C, allow me to examine the requirement $V(\arcsin \Gamma, 0) = 0$. We have that

$$\begin{aligned} V(\arcsin \Gamma, 0) &= -\cos(\arcsin \Gamma) - \Gamma \arcsin \Gamma + C \\ 0 &= -\sqrt{1-\Gamma^2} - \Gamma \arcsin \Gamma + C \end{aligned}$$

which implies that $C = \sqrt{1-\Gamma^2} + \Gamma \arcsin \Gamma + 2$ (I added the $+2$ term to account for positive definiteness for (x, y) in the neighbourhood of the fixed point).

Therefore the Liapunov function in the neighbourhood of the fixed point is

$$\boxed{V(x, y) = \frac{y^2}{2} - \cos(x) - \Gamma x + \Gamma \arcsin \Gamma + \sqrt{1-\Gamma^2} + 2.}$$

Since this Liapunov function exists, the linearized center is actually a nonlinear stable spiral, which is what I wanted to show.

(c) *Existence.* To show that the system has a unique, stable limit cycle for $\Gamma > 1$ as $t \to \infty$, allow me to first consider the bounds of the function defining the cycle, that is $y(x) = \sqrt{\frac{\Gamma - \sin x}{\alpha}}$ (I will obtain this more explicitly later in the problem). Note that it is continuous, and it is bounded below and above by two values $y_1 = \sqrt{\frac{\Gamma - 1 - k}{\alpha}}$ and $y_2 = \sqrt{\frac{\Gamma + 1_k}{\alpha}}$, respectively, for $k \ll 1$ to approximate the bounds being close to the minima and maxima of the cycle. Since the limit cycle is located in the $y > 0$ region, then for initial values above the cycle, we have that

$$\dot{y} = -\frac{\Gamma + 1 + k}{\alpha} \cdot \alpha - \sin x + \Gamma = -\sin x - 1 - k < 0,$$

hence every trajectory above this line has negative velocity $\dot{y} < 0$, so it is converging towards the limit cycle. Similarly, for initial values below the limit cycle,

$$\dot{y} = -\frac{\Gamma - 1 - k}{\alpha} \cdot \alpha - \sin x + \Gamma = -\sin x + 1 + k > 0,$$

so every trajectory below this line has positive velocity $\dot{y} > 0$, and therefore the limit cycle is stable. Every trajectory which enters the strip $y_1 \leq y \leq y_2$ will remain there. Since $y > 0$ in this whole strip, $\dot{x} > 0$, and thus trajectories will be flowing to the right as $t \to \infty$.

Uniqueness. Since $\Gamma > 0$, as noted before, no fixed points exist in the system and every trajectory will converge to a limit cycle as shown above. Now assume there are two different limit cycles $y_1(x)$ and $y_2(x)$ with $y_1(x) > y_2(x)$. Throughout one cycle, due to periodicity, the change in energy must be zero $\Delta E = 0$. Consider the energy function $E = \frac{y^2}{2} - \cos x$, where $y = y(x) = \sqrt{\frac{\Gamma - \sin x}{\alpha}}$. This implies that

$$\frac{dE}{dx} = y \cdot \frac{dy}{dx} + \sin x = y \cdot \frac{y'}{x'} + \sin x = y \frac{\Gamma - \sin x - \alpha y^2}{y} + \sin x = \Gamma - \alpha y^2.$$

The change in energy over one period is

$$\begin{aligned}\int_0^{2\pi} dE &= \int_0^{2\pi} \frac{dE}{dx}\,dx \\ &= \int_0^{2\pi} (\Gamma - \alpha y^2)\,dx \\ &= 0\end{aligned}$$

which implies that $\displaystyle\int_0^{2\pi} y^2\,dx = \frac{2\pi\Gamma}{\alpha}$. However, if $y_1(x) > y_2(x)$ by assumption, then $\displaystyle\int_0^{2\pi} y_1^2(x)\,dx > \int_0^{2\pi} y_2^2(x)\,dx$, which contradicts the fact that both integrals must be equivalent to $\dfrac{2\pi\Gamma}{\alpha}$. Therefore there is only one stable limit cycle and therefore it is unique.

(d) If $u = \frac{1}{2}\dot{x}^2$, then

$$\begin{aligned}\frac{du}{dt} &= \frac{d}{dt}\left[\frac{1}{2}\dot{x}^2\right] = \dot{x}\ddot{x} \\ &= \frac{du}{dx}\frac{dx}{dt} = \frac{du}{dx}\dot{x} \\ \implies \frac{du}{dx} &= \ddot{x}.\end{aligned}$$

(e) Part (d) implies that the equation of the pendulum is given by

$$\frac{du}{dx} + 2\alpha u + \sin x = \Gamma.$$

The limit cycle occurs as an attractive nullcline, that is, when $\dfrac{du}{dx} = 0$. This now implies that

$$2\alpha u = \Gamma - \sin x.$$

Solving for $u = u(x)$ then gives an explicit solution for the limit cycle in terms of x:

$$u(x) = \frac{\Gamma - \sin x}{2\alpha}.$$

To determine the explicit formula in terms of y and x, one may proceed by simply undoing the substitution:

$$\frac{1}{2}y^2 = \frac{\Gamma - \sin x}{2\alpha} \implies \boxed{y(x) = \sqrt{\frac{\Gamma - \sin x}{\alpha}},}$$

which is the result which I had noted earlier in the problem.

(f) I will note that for any $\Gamma > 1$, no fixed points or saddles exist. As Γ decreases, new fixed points are created under a homoclinic bifurcation along the limit cycle trajectory. This can be shown via the Jacobian matrices as derived earlier in the problem.

For the fixed point $(\pi - \arcsin\Gamma, 0)$, we have

$$J_1 = \begin{pmatrix} 0 & 1 \\ \sqrt{1-\Gamma^2} & 0 \end{pmatrix}$$

and for $(\arcsin\Gamma, 0)$,

$$J_2 = \begin{pmatrix} 0 & 1 \\ -\sqrt{1-\Gamma^2} & 0 \end{pmatrix}.$$

For J_2, I have already shown that the linearized center is actually a nonlinear stable spiral via the method of Liapunov functions. Notice that when $\Gamma = 1$, both of the Jacobian matrices are identical. As Γ decreases, the fixed points move apart and their stability changes. In terms of x, the bifurcation curve is simple given by the roots of the system along $y = 0$. That is, $\arcsin x$ and $\pi - \arcsin x$. The bifurcation diagram *for two fixed points* is then

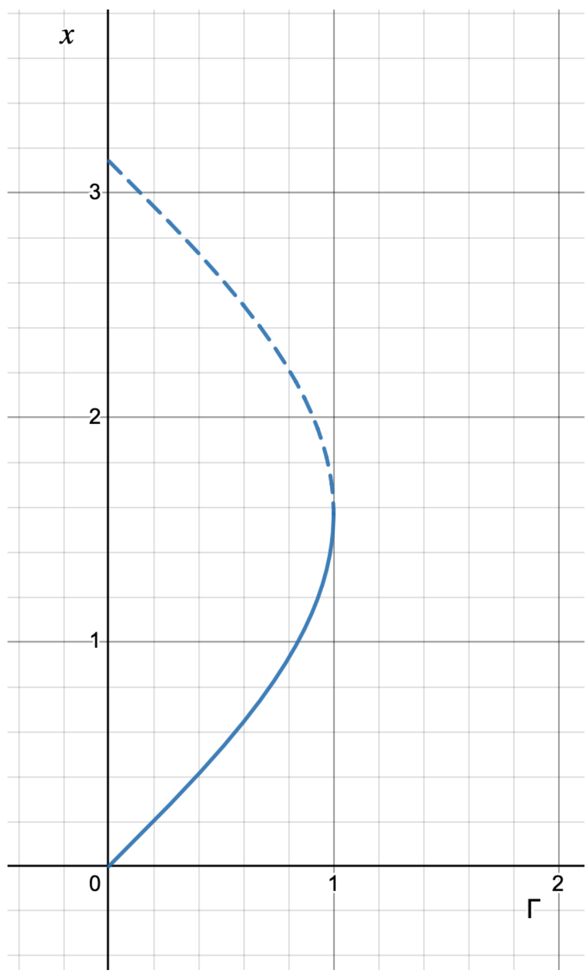


Figure 9: Bifurcation as Γ is varied in the nonlinear pendulum system. The dashed line represents the created saddle point, and the solid line the fixed stable spiral.

Note that this curve is independent of α, and hence α only works to affect the stability of each of these points. In various cases, α can radically change the behvaiour around the fixed points in the sense that for large α and $\Gamma < 1$, every solution will fall into a fixed point and a limit cycle does not exist. However for smaller α, the limit cycle will again appear in the dynamics. I shall note, just as Strogatz has in the textbook, that finding an explicit solution for the bifurcation curve $\Gamma(\alpha)$ requires more advanced methods such as Melnikov's methods. However, I will note that by my observation the behaviour of the dynamics as α and Γ are varied are equivalent to how they behave in the system with a linear damping term, such as in the Josephson Junction problem. Such a bifurcation diagram is given by Strogatz:

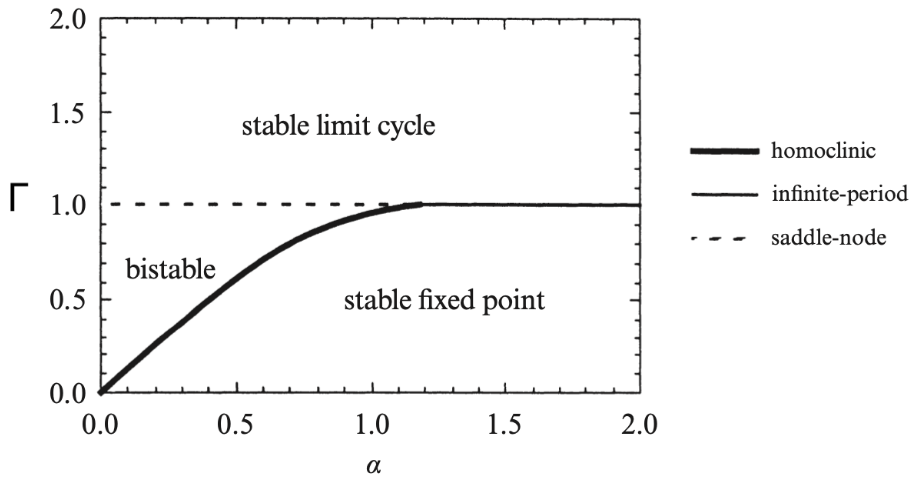
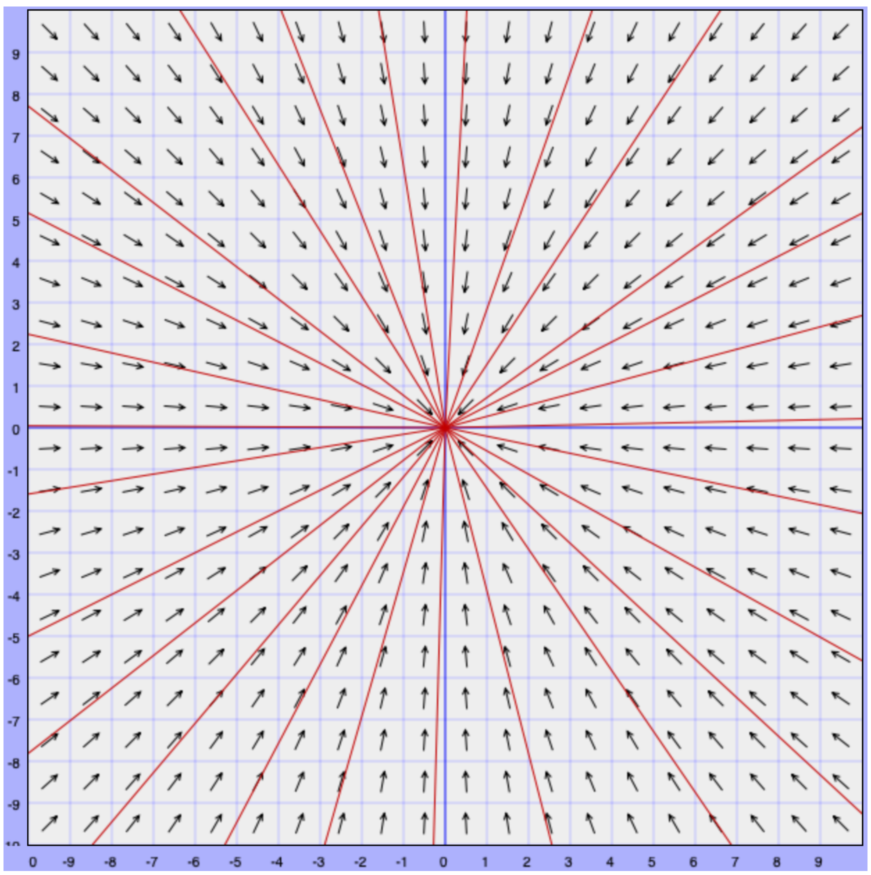


Figure 10: Taken from Strogatz (2018), Nonlinear Dynamics and Chaos, pp.275.

Finding the explicit equation which defines such a bifucation curve as this requires much more insight which I do not intend to note here.

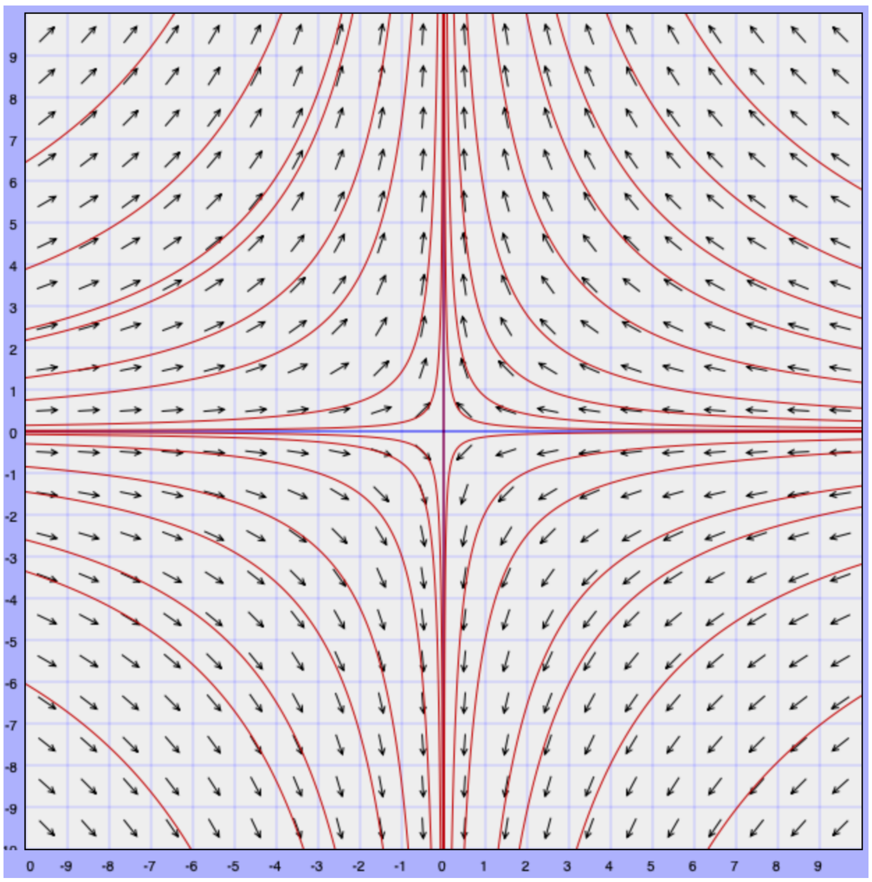
7.2.1

Consider the gradient system defined by the potential $V(x, y) = x^2 + y^2$, in which yields that $\begin{cases} \dot{x} = -2x \\ \dot{y} = -2y \end{cases}$. The linear matrix corresponding to this system is $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$, which has trace -4 and determinant 4. Since $(-4)^2 - 4(4) = 16 - 16 = 0$, the stability classification diagram gives that the origin is a star node or a degenerate node. It may suffice to compute the eigenvectors of the matrix. It is easy to see that they are actually any linear combination of x and y, since the system is uncoupled. Therefore we obtain a stable star node at the origin, since both eigenvalues are negative:



7.2.2

We now examine the potential function $V(x, y) = x^2 - y^2$, which relates to a saddle point in $\mathbb{R}^3$−space if you recall from multivariable calculus. We find that the corresponding system is $\begin{cases} \dot{x} = -2x \\ \dot{y} = 2y \end{cases}$, with matrix $\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$. Here, the determinant is -4 and the trace is 0, which directly correlates to a saddle node in phase space at the origin. Once again, the system is uncoupled, so the eigenvectors are just $\boldsymbol{v}_1 = (1, 0)$ and $\boldsymbol{v}_2 = (0, 1)$.

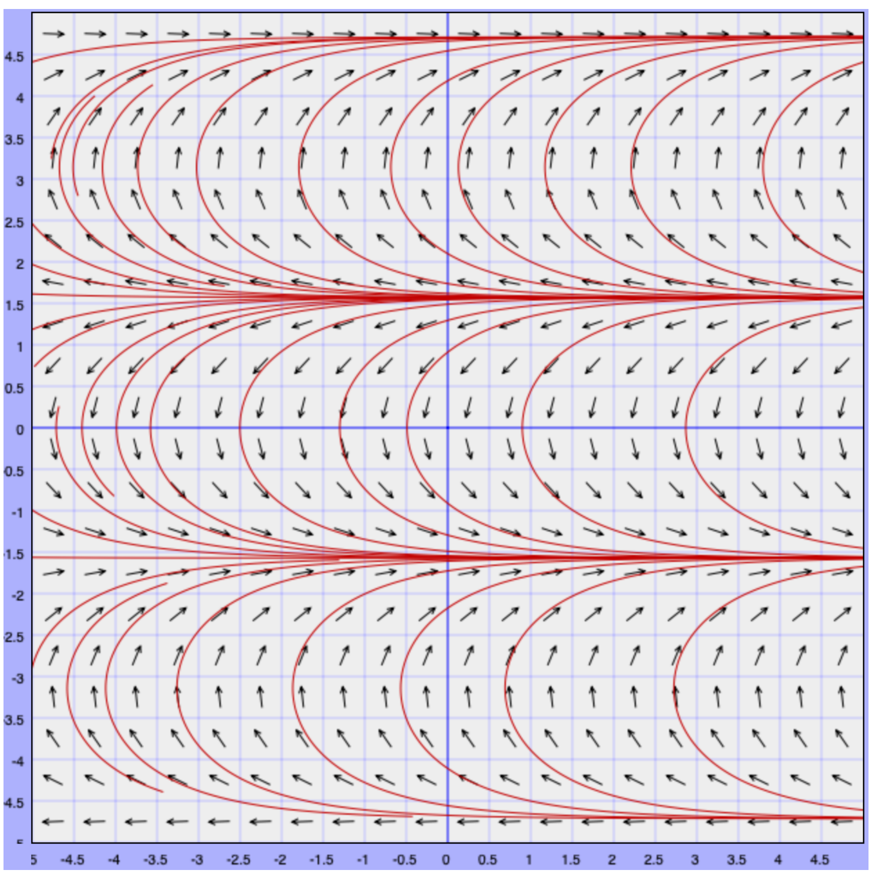


7.2.3

Now consider the potential given by $V(x,y) = e^x \sin y$. By taking the gradient, we find that the system is $\begin{cases} \dot{x} = -e^x \sin y \\ \dot{y} = -e^x \cos y \end{cases}$, which has no fixed points. Note that the system only has nullclines, and they are given only along the y-axis because e^x is never zero. This system primarily has two nullclines, given whenever $\dot{y} = 0$, which implies that $y = \pm\dfrac{\pi}{2}$. The x-direction has no nullclines once again, because e^x is never zero.

We may proceed to now determine the stability of these nullclines and henceforth plot the phase diagram. Note that for values $0 < y < \dfrac{\pi}{2}$, we have that $0 < \sin y < 1$, and $e^x > 0$, which implies that $\dot{x} < 0$ in this region; flows are to the left. For values $-\dfrac{\pi}{2} < y < 0$, then $-1 < \sin y < 0$ so $\dot{x} > 0$; flows are to the right. In a similar fashion, for $0 < y < \dfrac{\pi}{2}$, then $0 < \cos y < 1$, hence $\dot{y} < 0$; flows are downwards. Lastly, $-\dfrac{\pi}{2} < y < 0$, then $-1 < \cos y < 0$; flows are still downwards.

These results imply that the $y = \dfrac{\pi}{2}$ nullcline is unstable, and the $y = -\dfrac{\pi}{2}$ cline is stable. Under the periodic motion provided to us by the $\sin y$ and $\cos y$ term in the cycle, this implies that every nullcline $y = \dfrac{\pi}{2} + 2\pi k$, $k \in \mathbb{Z} \cup \{0\}$, is unstable, and every $y = -\dfrac{\pi}{2} + 2\pi k$ is stable. Therefore the phase portrait appears as



8.1.9

Consider the system given by the equation of motion $\ddot{x} + b\dot{x} - kx + x^3 = 0$. One may rewrite this second order system as a first order coupled system of differential equations, such as

$$\begin{cases} \dot{x} = y \\ \dot{y} = -by + kx - x^3 \end{cases}$$

Since b and k are any-valued real parameters, the number of roots of each equation may change depending on b and k, hence a bifurcation occuring. Allow me to focus on the equation $\dot{y} = -by + kx - x^3$. Factoring this equation gives that $\dot{y} = -by + x(k - x^2)$.

Notice that every fixed point of the system must lie on the $y = 0$ axis, else $\dot{x} \neq 0$. Furthermore, the solutions of the polynomial $x(k - x^2)$ are $x = 0$ and $x = \pm\sqrt{k}$, and therefore all of our fixed points are given as

$$(0, 0), \qquad (\pm\sqrt{k}, 0).$$

To further build on our notion of the fixed points of the system, allow me to consider the stability of each fixed point. By linearizing the system about each of the fixed points beginning with the Jacobian, we have that

$$J = \begin{pmatrix} 0 & 1 \\ k - 3x^2 & -b \end{pmatrix} \Bigg|_{(x^*, y^*)}$$

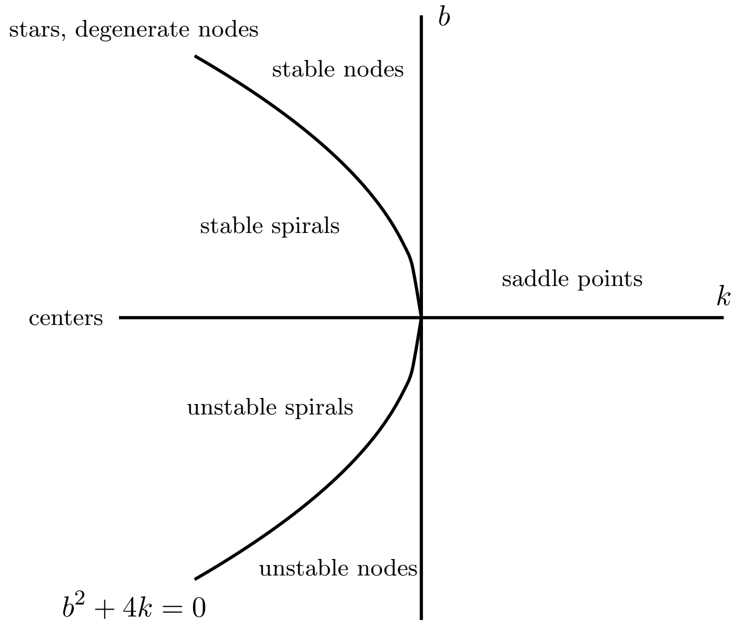
which, when evaluated at each of our fixed points, give

$$J_0 = \begin{pmatrix} 0 & 1 \\ k - 3x^2 & -b \end{pmatrix} \Bigg|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ k & -b \end{pmatrix}$$
$$J_\pm = \begin{pmatrix} 0 & 1 \\ k - 3x^2 & -b \end{pmatrix} \Bigg|_{(\pm\sqrt{k},0)} = \begin{pmatrix} 0 & 1 \\ -2k & -b \end{pmatrix}.$$

First, examine the $(0, 0)$ point. First, note that this fixed point will always exist for any value of k and b, since is it trivially at the origin. The trace of J_0 is $-b$ and determinant $-k$. Going off of the stability diagram for linearized matrices as studied in class, we have that

- For $k > 0$, we will always obtain a saddle node, for any value of b.
- For $k < 0$, the the type of node is only dependent on b. $b = 0$ implies a center.
- $k < 0$, $b > 2\sqrt{|k|}$ are stable nodes.
- $k < 0$, $2\sqrt{|k|} > b > 0$ are stable spirals.
- $k < 0$, $0 > b > -2\sqrt{|k|}$ are unstable spirals.
- $k < 0$, $-2\sqrt{|k|} > b$ are unstable nodes.
- $k < 0$, $b = |2\sqrt{|k|}|$ are stars and degenerate nodes; unstable for $b > 0$ and stable for $b < 0$.

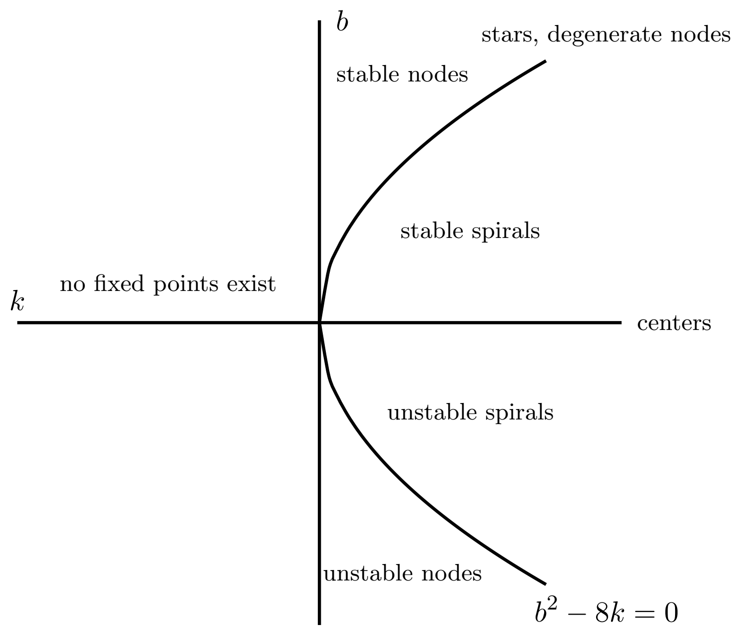
The corresponding stability diagram for $(0, 0)$ is then



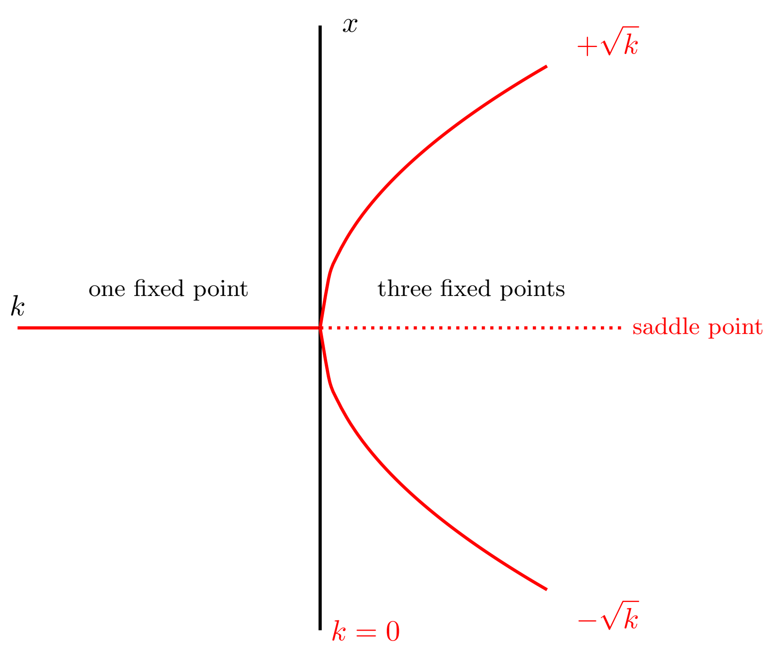
We may now apply a similar method to determining the stability of the two branches which are created throughout the duration of the pitchfork bifurcation. Note that, the fixed points $(\pm\sqrt{k}, 0)$, are only existent when $k > 0$ (which relates to the origin being classified as a saddle node). The trace of $J_{\pm}$ is again $-b$, while the determinant is $2k$. Since we are only examining values of $k > 0$, then it suffices just to look at b. For $k > 0$ at the $(\pm\sqrt{k}, 0)$ fixed points, we have (going off of the previous arguments):

- $b = 0$ implies centers
- $b > 2\sqrt{2k}$ implies stable nodes
- $2\sqrt{2k} > b > 0$ are stable spirals
- $0 > b > -2\sqrt{2k}$ are unstable spirals
- $b < -2\sqrt{2k}$ are unstable nodes
- $b = | \pm 2\sqrt{2k}|$ give stars and degenerate nodes

The corresponding stability diagram is similar to the one previous drawn:



Lastly, one may illustrate the bifurcation diagram in the (k, x) plane, since the location of every fixed point is independent of b:



This completes our analysis on this problem.