PHY460 Problem Set 4 — Due December 7, 17:00

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1*.

For this problem, I will be consistently revisiting Liouville's theorem for Hamiltonian Systems. Simply stated, let D be a bounded and measurable subset of $\mathbb{R}^2$, and let $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ be a vector field in $\mathbb{R}^2$. Let $\varphi_t(D)$ be the image of D under the flow (trajectory) at time t, and let $A = \displaystyle\int_{\varphi_t(D)} dA$ be the area of D under this flow at time t. If D is bounded and measurable, then the rate of change of area is given by

$$\dot{A} = \int_{\varphi_t(D)} \boldsymbol{\nabla} \cdot \boldsymbol{f}(\boldsymbol{x})\, dA.$$

One corollary of this theorem is that Hamiltonian systems $\left(\dot{x} = \dfrac{\partial H}{\partial y},\ \dot{y} = -\dfrac{\partial H}{\partial x}\right)$ are area-preserving in phase space. I took this theorem from planetmath.org, with its proof included.

(a) Consider the system as given $\begin{cases} \dot{x} = -y(1+x^2+y^2) \\ \dot{y} = x(1+x^2+y^2) \end{cases}$. Note that the divergence of this system is

$$\boldsymbol{\nabla} \cdot \big(-y(1+x^2+y^2), x(1+x^2+y^2)\big) = -2xy + 2xy = 0,$$

hence by Liouville's theorem for Hamiltonian systems, this system is area preserving. We obtain that $\dot{A} = 0$, hence $\dfrac{1}{A}\dfrac{dA}{dt} = 0$. One may check more explicity by integration that $\dot{x} = \dfrac{\partial H}{\partial y} \implies H = -\dfrac{1}{2}y^2x^2 - \dfrac{1}{4}y^4 - \dfrac{1}{2}y^2 + f(x)$, and $\dot{y} = -\dfrac{\partial H}{\partial x} \implies H = -\dfrac{1}{4}x^4 - \dfrac{1}{2}x^2y^2 - \dfrac{1}{2}x^2 + g(y)$, which implies that the Hamiltonian of the system does indeed exist and is

$$H = -\frac{1}{2}x^2y^2 - \frac{1}{2}(x^2+y^2) - \frac{1}{4}(x^4+y^4)$$

Therefore this system is area preserving.

(b) To transform the system into polar coordinates, we can consider the variable substitutions $x = r\cos\theta$ and $y = r\sin\theta$. The equations of the system then become

$$\begin{cases} \dot{r}\cos\theta - r\sin\theta\dot{\theta} = -r\sin\theta(1+r^2) \\ \dot{r}\sin\theta + r\cos\theta\dot{\theta} = r\cos\theta(1+r^2). \end{cases}$$

This implies that $\dot{r} = \dfrac{r\sin\theta(\dot{\theta} - 1 - r^2)}{\cos\theta}$, hence $r\sin^2\theta(\dot{\theta} - 1 - r^2) + r\cos^2\theta\dot{\theta} = r\cos^2\theta(1+r^2)$. It follows that

$$r\sin^2\theta\dot{\theta} + r\cos^2\theta\dot{\theta} = r\cos^2\theta(1+r^2) + r\sin^2\theta(1+r^2) \implies r\dot{\theta} = r(1+r^2),$$

whic directly implies that

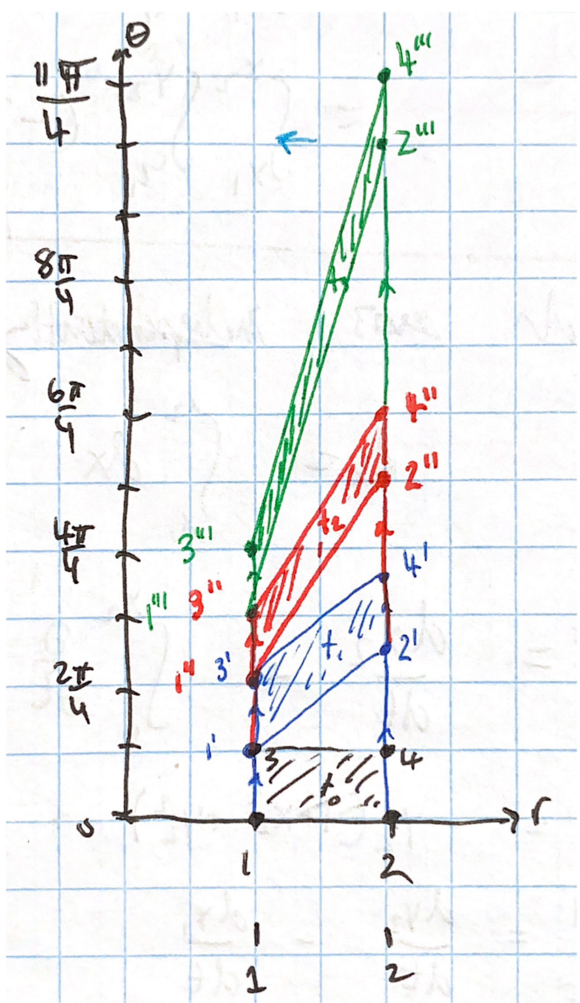
$$\dot{\theta} = 1 + r^2 \implies \dot{r} = 0.$$

Therefore the system in polar coordinates is $\begin{cases} \dot{r} = 0 \\ \dot{\theta} = 1 + r^2 \end{cases}$.

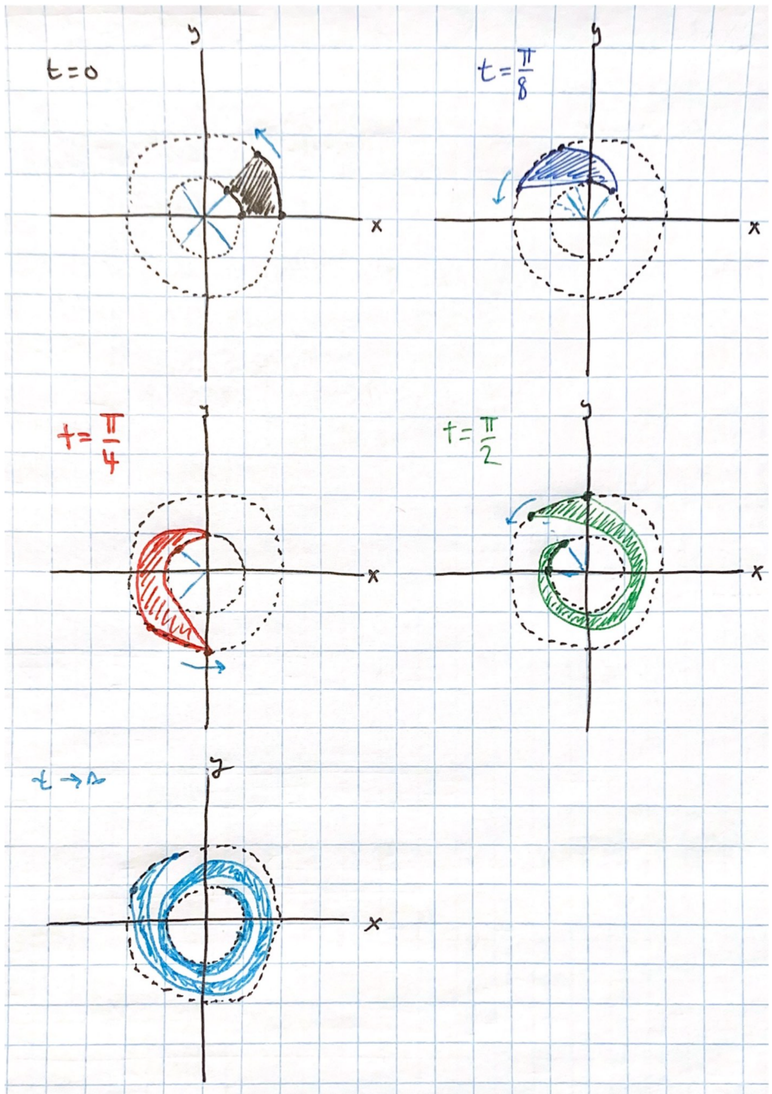
(c) Although the system written in cartesian form is complicated to solve explicity, the system written in polar coordinates gives simply elegant results for the explicit equations which depict the motion. $\dot{r} = 0$ implies that $r = r_0$ for a constant initial value $r_0 \geq 0$. This implies that $\dot{\theta} = 1 + r_0^2$. Seperating variables and integrating yields $\theta(t) = (1 + r_0^2)t + \theta_0$ for some initial release angle θ_0. It is clear to see now that phase space area does not change, but rather just distorts.
Consider a rectangle in polar coordinates given by the vertices $(1, 0), (2, 0), (1, \pi/4), (2, \pi/4)$. The equations of motion will then determine the distortion of this box over time:

$$\begin{aligned}
(1,0) \longrightarrow & r = 1,\ \theta(t) = 2t. \\
& \implies \theta(\pi/8) = \pi/4,\ \ \theta(\pi/4) = \pi/2,\ \ \theta\pi/2 = \pi \\
(2,0) \longrightarrow & r = 2,\ \theta(t) = 5t. \\
& \implies \theta(\pi/8) = 5\pi/8,\ \ \theta(\pi/4) = 5\pi/4,\ \ \theta\pi/2 = 5\pi/2 \\
(1,\pi/4) \longrightarrow & r = 1,\ \theta(t) = 2t + \pi/4. \\
& \implies \theta(\pi/8) = \pi/2,\ \ \theta(\pi/4) = 3\pi/4,\ \ \theta\pi/2 = 5\pi/4 \\
(2.\pi/4) \longrightarrow & r = 2,\ \theta(t) = 5t + \pi/4. \\
& \implies \theta(\pi/8) = 7\pi/8,\ \ \theta(\pi/4) = 6\pi/4,\ \ \theta\pi/2 = 11\pi/4.
\end{aligned}$$

In terms of a diagram, with $t_0 = 0, t_1 = \pi/8, t_2 = \pi/4$, and $t_3 = \pi/2$, we have that



(d) In cartesian coordinates, this box represents a region in polar coordinates. As time increases, the points on the radius line $r = 2$ have a larger tangential velocity, about 2 radians per second (if we use such units). As time increases, the shape will distort into a spiral, with each line not intersecting, since the tangential equation of motion $\theta = (1 + r_0^2)t + \theta_0$ is in the form of an Archimedean spiral. One can see from the previous diagram that the area of the shape remains constant, since the base and height do not change.



(e) Altering the dynamics to the system

$$\begin{cases} \dot{x} = -y(1 + x^2 + y^2) + x(1 - x^2 - y^2) \\ \dot{y} = x(1 + x^2 + y^2) + y(1 - x^2 - y^2) \end{cases}$$

now no longer leaves the system in an elegant Hamiltonian form, since $\nabla \cdot \mathbf{f}(\mathbf{x}) = 2(1 - 2x^2 - 2y^2) \neq 0$. This implies that the flow now distorts and changes the volume of measurable subsets in $\mathbb{R}^2$. Since the trajectories exist, let me call them $x(t)$ and $y(t)$ in cartesian form. Consider a set

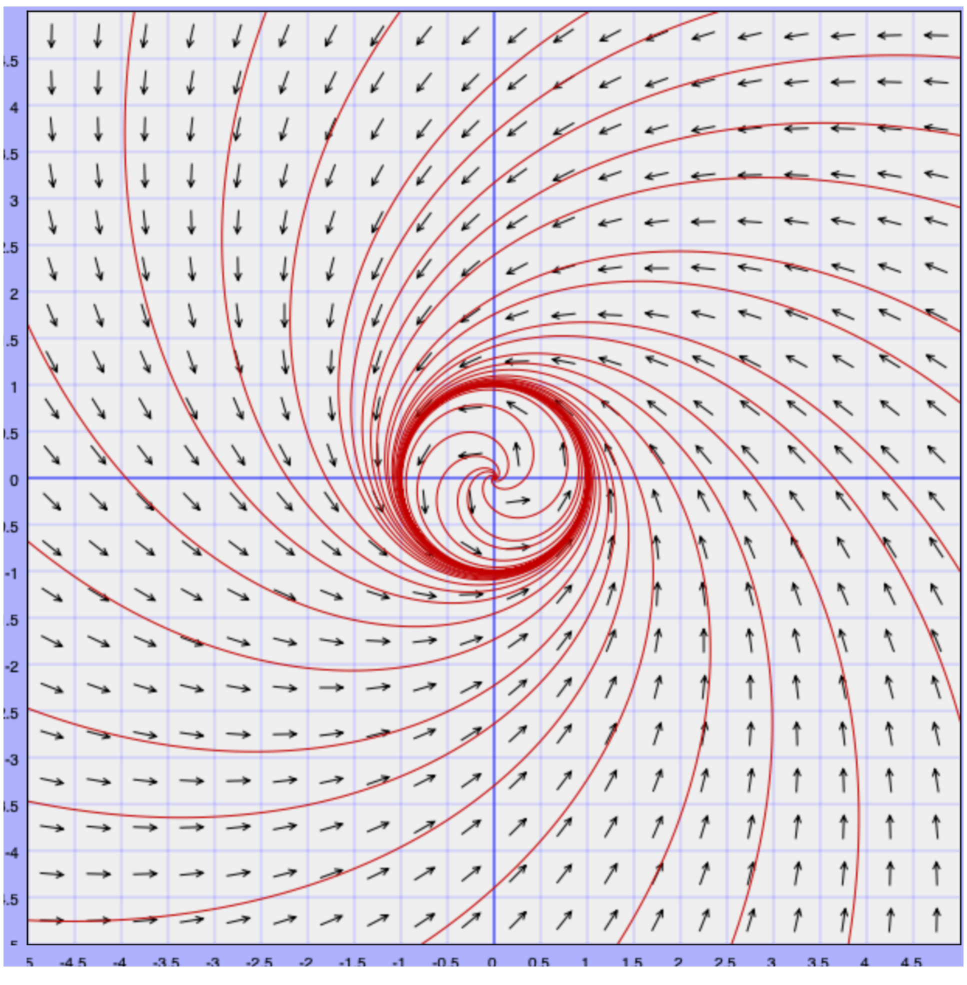
D under the influence of this flow. The area after a time t is given by $A = \int_{\varphi_t(D)} dA$, where $\varphi_t(D) = \varphi(x_D(t), y_D(t))$ is the image of D under the flow at time t. Then, the rate of change of phase space area is

$$\frac{1}{A}\frac{dA}{dt} = \left[\int_{\varphi_t(D)}\right]^{-1} \cdot 2\int_{\varphi_t(D)} (1 - 2x^2 - 2y^2)\, dxdy.$$

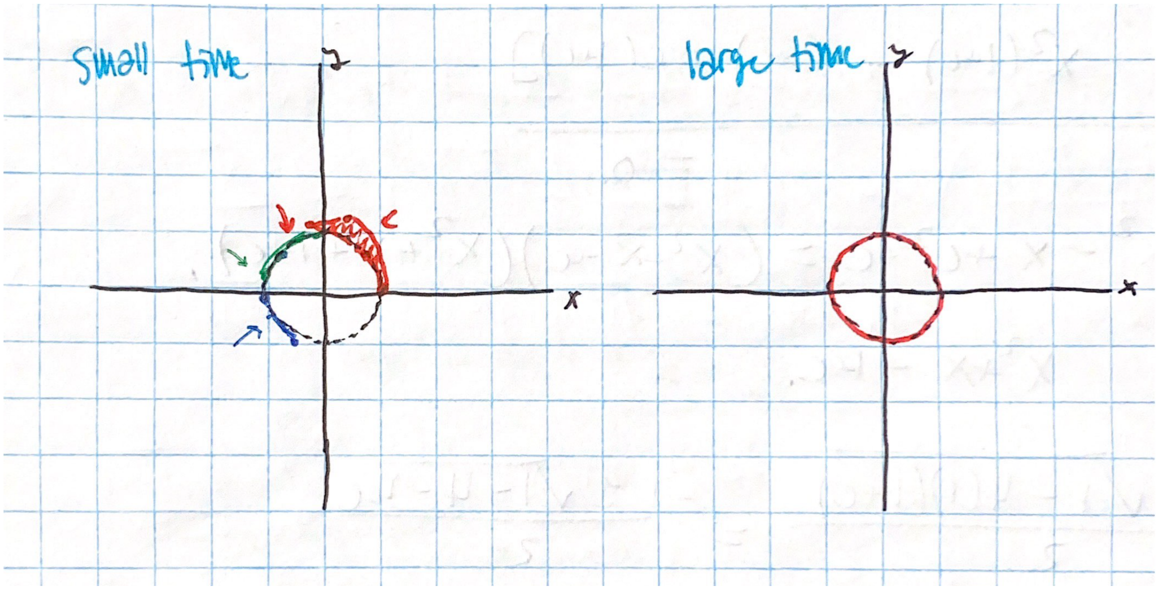
This may be easier to interpret for a polygon of four vertices. Let $(x_1(t), y_1(t)), (x_2(t), y_1(t)), (x_1(t), y_2(t)), (x_2(t), y_2($ be its vertices after a time t. Then $A = \int_{x_1(t)}^{x_2(t)} \int_{y_1(t)}^{y_2(t)} dydx = (x_2(t) - x_1(t))(y_2(t) - y_1(t))$ is the area of the box after a time t. The change in phase space area is then

$$\begin{aligned} \frac{1}{A}\dot{A} &= \frac{2}{(x_2(t) - x_1(t))(y_2(t) - y_1(t))} \int_{x_1(t)}^{x_2(t)} \int_{y_1(t)}^{y_2(t)} (1 - 2x^2 - 2y^2)\, dydx \\ &= 2 - \frac{4}{3}\frac{1}{(x_2(t) - x_1(t))(y_2(t) - y_1(t))} \left[(x_2(t))^3 - (x_1(t))^3\right] \left[(y_2(t))^3 - (y_1(t))^3\right]. \end{aligned}$$

Considering the box from part (d) of this problem, it is easy to see that $\dot{A}$ is negative for this system, hence the net area is decreasing in phase space. Numerically plotting the phase portrait gives the attractor to be a limit cycle at radius 1, with trajectories curling counterclockwise since $\dot{\theta} > 0$.



Eventually, every shape will be crushed into a line (or circle of radius 1, really) as $t \to \infty$.



Algebraiaclly, this makes sense, since $\dot{A}$ is negative, so every measureable domain will eventually be crushed into a one-dimensional arealess object.

7.2.9

(a) Consider the system $\begin{cases} \dot{x} = y + x^2 y \\ \dot{y} = -x + 2xy \end{cases}$. Note that the curl of a gradient is always zero, so if the curl of this system is zero, then it is a gradient system. In $\mathbb{R}^2$, we have that

$$\text{curl}(y + x^2 y,\, -x + 2xy) = \frac{\partial}{\partial x}[-x + 2xy] - \frac{\partial}{\partial y}[y + x^2 y] = -2 + 2y - x^2 \neq 0,$$

hence this system is not a gradient system.

(b) We may apply a similar calculation as in the previous part (a). Note that the system $\begin{cases} \dot{x} = 2x \\ \dot{y} = 8y \end{cases}$ is uncoupled, hence it can be easily represented as a gradient system as a sum in individual uncoupled, seperable components

$$V(x, y) = -x^2 - 4y^2,$$

hence $-\boldsymbol{\nabla} V(x, y) = (2x, 8y)$, as desired.

(c) Lastly, consider the system given by $\begin{cases} \dot{x} = -2xe^{x^2+y^2} \\ \dot{y} = -2ye^{x^2+y^2} \end{cases}$. This system is clearly a gradient system, because it is seperable. Note that $\dfrac{\partial}{\partial x}[e^{x^2+y^2}] = e^{y^2}\dfrac{\partial}{\partial x}[e^{x^2}] = 2xe^{x^2+y^2}$, and equivalently $\dfrac{\partial}{\partial y}[e^{x^2+y^2}] = 2ye^{x^2+y^2}$. Therefore the gradient system is of the potential

$$V(x, y) = e^{x^2+y^2},$$

hence $-\boldsymbol{\nabla} V(x, y) = (-2xe^{x^2+y^2}, -2ye^{x^2+y^2})$, thus this system is a gradient system.

8.1.13

In this problem we consider the system which models the laser model

$$\begin{cases} \dot{n} = GnN - kn \\ \dot{N} = -GnN - fN + p \end{cases}$$

where n is the number of atoms, N the number of photons in the laser field, G the gain coefficient, k the decay rate, d the spontaneous decay rate, and p the pump strength.

(a) To nondimensionalize, it may be useful to first determine the units of each of the parameters. $\dot{n}$ has units $\#/T$, hence kn and GnN should also have these units. We find that k has units of $1/T$ and G has units $1/\#T$. With this, we must also have that f has units of $1/T$ and p has units of $\#/T$.

To nondimensionalize the system, it suffices to first find combinations which lead to nondimensional quantities. Obtaining nondimensional terms for n and N then allow us to introduce nondimensional parameters both on the left and right hand side of the equation.

First note that factoring out a k in $\dot{x}$ yields the nondimensional parameter $\frac{G}{k}n$. In the same way, in the equation for $\dot{y}$, factoring out a p yields the nondimensional parameter $\frac{f}{p}N$. Similarly, introducing the nondimensional time ft allows us the rewrite the nondimensional equations. Therefore, let

$$x = \frac{G}{k}n, \qquad y = \frac{f}{p}N \qquad \tau = ft.$$

Therefore, we obtain our system as

$$\begin{cases} \dfrac{dk}{G}\dfrac{dx}{d\tau} = \dfrac{kp}{f}xy - \dfrac{k^2}{G}x \\ p\dfrac{dy}{d\tau} = -\dfrac{kp}{f}xy - py + p \end{cases}$$

Each of these equations has dimensions of $\#/T$, hence multiplying each side by its respective conjugate yields the nondimensional equations:

$$\begin{cases} \dfrac{dx}{d\tau} = \dfrac{Gp}{f^2}xy - \dfrac{k}{f}x \\ \dfrac{dy}{d\tau} = -\dfrac{k}{f}xy - y + 1 \end{cases}$$

This suggests I introduce the nondimensional parameters $a = \frac{k}{f}$ and $b = \frac{Gp}{f^2}$. Here, a must be strictly positive since $k, f > 0$ and b can have any sign, since p can have any sign. Therefore

$$\begin{cases} \dot{x} = bxy - ax \\ \dot{y} = -axy - y + 1 \end{cases}$$

is the nondimensional system.

(b) The fixed points of the nondimensional system are given by $(\dot{x}, \dot{y}) = (0, 0)$ at (x^*, y^*). Factoring equations in the system give

$$\begin{cases} \dot{x} = x(by - a) \\ \dot{y} = -y(ax + 1) + 1 \end{cases}$$

so $\dot{x} = 0$ when $x = 0$ or $y = \dfrac{a}{b}$. If $x = 0$, then $\dot{y} = 0 \iff y = 1$, so one fixed point is $(0, 1)$. If $y = \dfrac{a}{b}$, then $\dot{y} = 0 \iff x = \dfrac{b-a}{a^2}$. Therefore the fixed points of the system are]

$$(x^*, y^*) = (0, 1), \ \left(\frac{b-a}{a^2}, \frac{a}{b}\right).$$

The first observation that one should make is that the fixed point $(0, 1)$ is independent of a and b and hence will always exists as a fixed point. The second fixed point only exists for $a, b \neq 0$ and $a \neq b$, else we obtain underdefined values or the fixed point $(0, 1)$ again. Classifying the fixed points follows by linearization, hence the Jacobian of the system is

$$J = \begin{pmatrix} by - a & bx \\ -ay & -ax - 1 \end{pmatrix} \Bigg|_{(x^*, y^*)}$$

and evaluated at the fixed points is

$$J_1 = \begin{pmatrix} b - a & 0 \\ -a & -1 \end{pmatrix}$$

$$J_2 = \begin{pmatrix} 0 & \dfrac{b}{a^2}(b - a) \\ -\dfrac{a^2}{b} & -\dfrac{b}{a} \end{pmatrix}.$$

Consider the fixed point $(0, 1)$ and J_1. Its trace is $b - a - 1$ and has determinant $a - b$. Note that, according to the stability diagram for matrices, $a - b < 0$ or $a < b$ implies that $(0, 1)$ is a saddle point. The eigenvalues of the matrix are the diagonal entries, since the matrix is lower-triangular. $\lambda_{1,2} = -1, b - a$. Both of the eigenvalues are real, so no spirals can occur. When $b < a$, we obtain a stable node at $(0, 1)$. Note the polynomial $(b - a - 1)^2 - 4(a - b) = 0$ corresponds to a line in the (a, b) plane, and solutions to this polynomial yield degenerate nodes. No centers can occur. Lastly, when $a = b$, only one eigendirection exists since the only eigenvalue is $\lambda = -1$.

Now consider the fixed point $\left(\dfrac{b-a}{a^2}, \dfrac{a}{b}\right)$ and J_2. The trace of J_2 is $-\dfrac{b}{a}$ and determinant $b - a$. For any value $a > b$, we obtain a saddle node. When $a = b$, the fixed point is the same as $(0, 1)$ and depicts a nonisolated fixed point. Now consider the eigenvalues of the matrix:

$$\lambda_{1,2} = \frac{-b/a \pm \sqrt{b^2/a^2 - 4(b - a)}}{2}.$$

Whenever $\dfrac{b^2}{a^2} - 4(b - a) < 0$, we obtain complex eigenvalues, so spirals exist. When $\dfrac{b^2}{a^2} - 4(b - a) = 0$, degenerate nodes, and $\dfrac{b^2}{a^2} - 4(b - a) > 0$ we obtain two distict eigenvalues which depict a stable node for $a < b$. This completes the classification of the fixed points.

(c) The following phase plots show the previous fixed points and they stabilities as described. The fixed points $(1) \equiv (0, 1)$ and $(2) \equiv \left(\dfrac{b-a}{a^2}, \dfrac{a}{b} \right)$ are circled in blue.

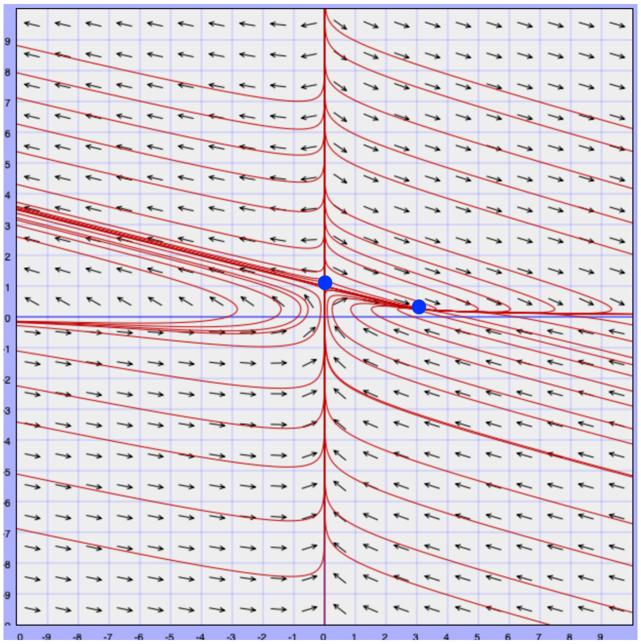


Figure 1: $a = 1, b = 4$. $b > a$, so (1) is saddle and (2) is stable node

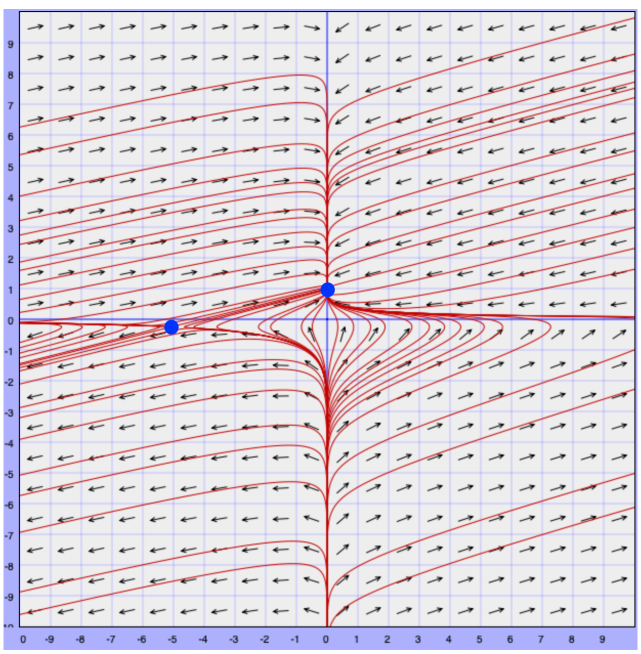


Figure 2: $a = 1, b = -4$. $b < a$, so (1) is a stable node and (2) is a saddle.

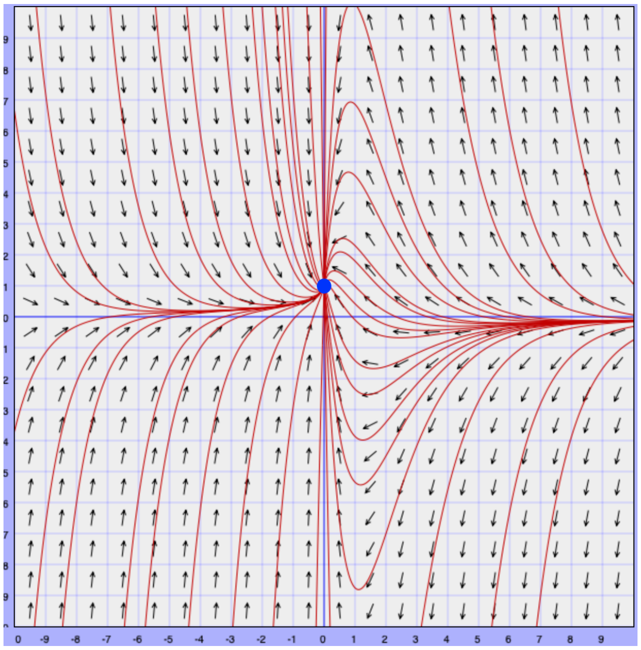


Figure 3: $a = 1, b = 0$. $b = 0$, so (2) is nonexistent, and (1) is a stable node.

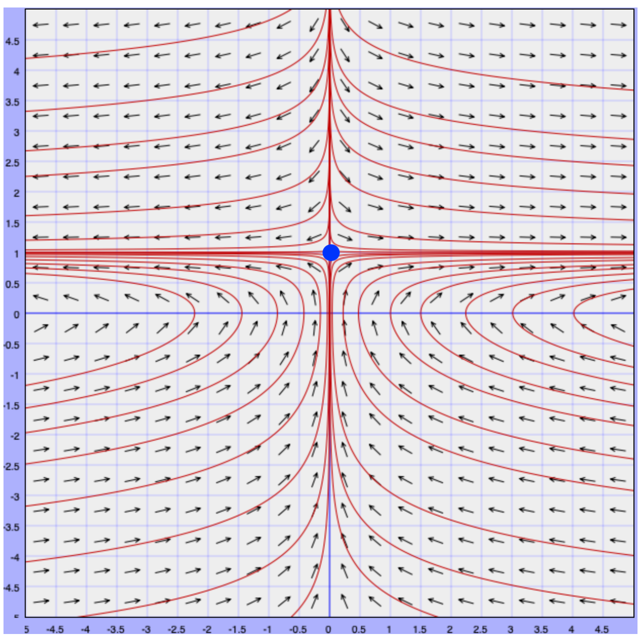


Figure 4: $a = 0, b = 2$. $a = 0$, so (2) is nonexistent, while (1) depicts a saddle point since $b > a$.

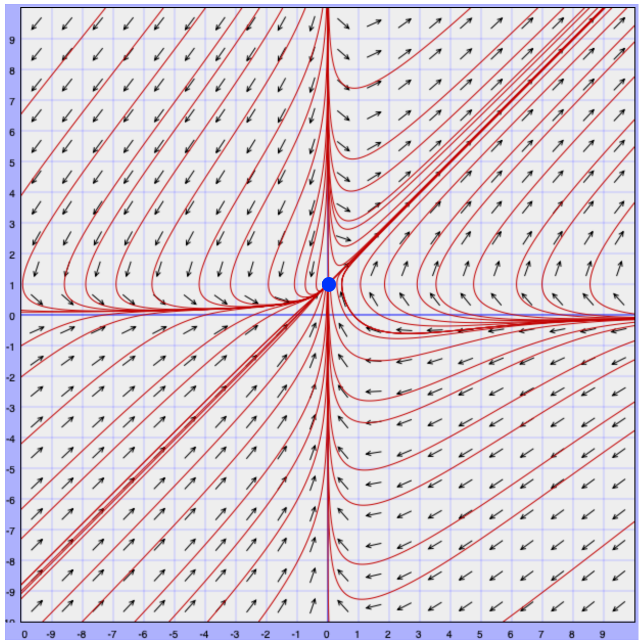


Figure 5: $a = 1, b = 1$. $a = b$, so (2) is nonexistent, while (1) depicts a nonisolated fixed point.

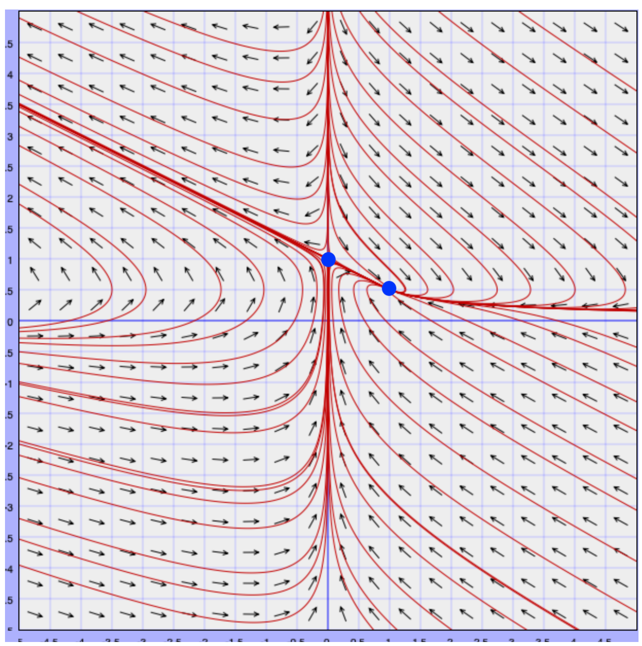


Figure 6: $a = 1, b = 2$. Here, $b > a$ so (1) is a saddle, while (2) is a degenerate node.

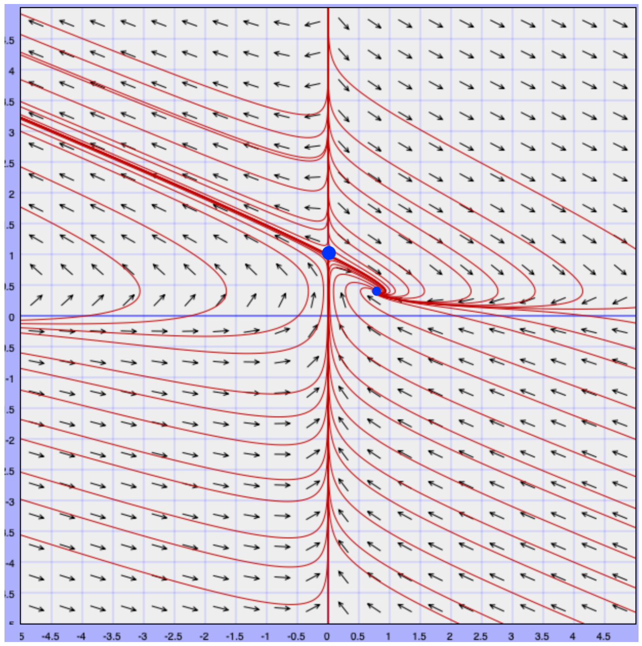
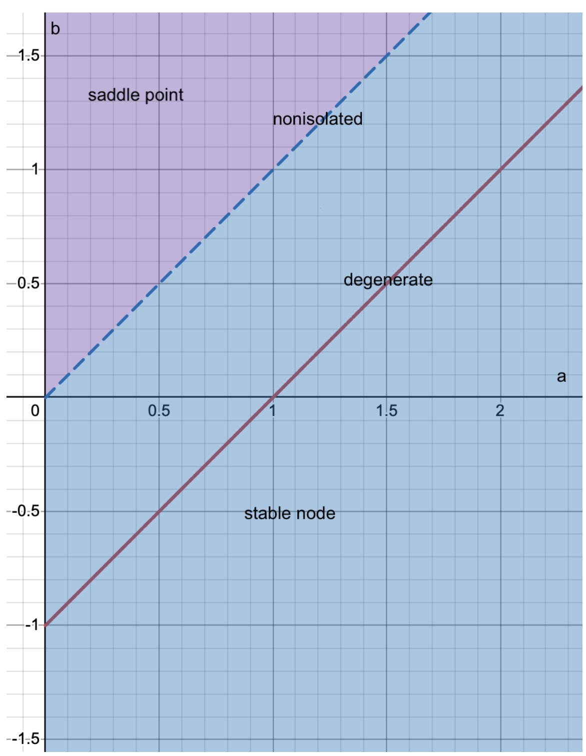
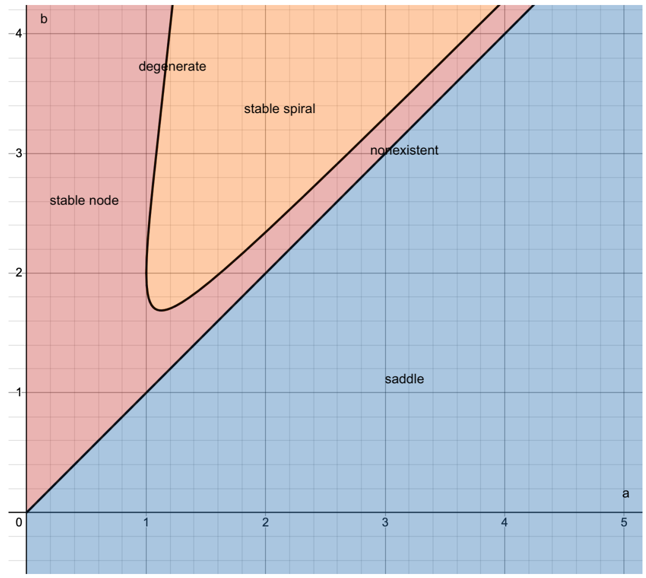


Figure 7: $a = 2, b = 5$. $b > a$, so (1) again is a saddle, however (2) is a stable spiral.

(d) The stability diagrams are each plotted for the fixed points (1) and (2) according to their classification:





Observe the line in each stability plot $a = b$ (or the $y = x$ line). When $a - b > 0$, notice that (2) has a saddle point. As b is varied, once $a - b < 0$, the saddle stability bifurcates over to the fixed point (1). By definition, a transcritical bifucation has occured. In each region, flip bifucations affect each indivial point as specified on their stability plots. Moreoever, all stability of each point seems to both 'swap' along the $b = a$ line, hence a transcritical bifurcation occurs.

9.2.1

(a) Consider the Lorenz system, as given by

$$\begin{cases} \dot{x} = \sigma y - \sigma x \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases}$$

where σ, r, b are the parameters. We first begin by linearizing the system by taking the Jacobian:

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}$$

evaluated along the fixed point curves (x^*, y^*, z^*). The curves $C_\pm$ at which fixed points occur are given by $C_\pm = x^* = y^* = \pm\sqrt{b(r-1)}$ while $z^* = r - 1$. At $C_\pm$, the linearized system becomes

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \mp\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b \end{pmatrix},$$

and we wish to find the eigenvalues of this linearized matrix. First, consider the characteristic polynmial:

$$\begin{aligned} \text{char}(J) &= \det \begin{pmatrix} -\sigma - \lambda & \sigma & 0 \\ 1 & -1 - \lambda & \mp\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b - \lambda \end{pmatrix} \\ &= (-\sigma - \lambda)(-1 - \lambda)(-b - \lambda) + \sigma(-b(r-1)) + 0 - 0 - (-b(r-1))(-\sigma - \lambda) - \sigma(-b - \lambda) \\ &= -\sigma b - \sigma b\lambda - b\lambda - b\lambda^2 - \sigma\lambda - \sigma\lambda^2 - \lambda^2 - \lambda^3 - 2\sigma br + 3\sigma b - \lambda br + \lambda b + \lambda\sigma \\ &= -\lambda^3 - \lambda^2(b + \sigma + 1) - b\lambda(\sigma + r) - 2\sigma b(r-1) \\ 0 &= \lambda^3 + \lambda^2(b + \sigma + 1) + \lambda b(\sigma + r) + 2\sigma b(r-1) \end{aligned}$$

which is the characteristic polynomial of the linearized system about $C_\pm$. Note that $C_\pm$ have the same characteristic equation, hence the same eigenvalues.

(b) Suppose we seek solutions of the purely imaginary form $\lambda = i\omega$ when $r = r_H = \sigma\left(\dfrac{\sigma + b + 3}{\sigma - b - 1}\right)$. Then, substitution gives the characteristic equation to be

$$0 = -i\omega^3 - \omega^2(b + \sigma + 1) + i\omega b(\sigma + r_H) + 2\sigma b(r_H - 1).$$

If this were true, we would require the real and imaginary components both to be zero:

$$\begin{aligned} 0 &= -\omega^3 + \omega b(\sigma + r_H) \\ 0 &= -\omega^2(b + \sigma + 1) + 2\sigma b(r_H - 1). \end{aligned}$$

I may further elaborate and explicity write out of the form in terms of r_H:

$$0 = -\omega^3 + \omega b \left(\sigma + \sigma \left(\frac{\sigma + b + 3}{\sigma - b - 1} \right) \right)$$
$$0 = -\omega^2 (b + \sigma + 1) + 2\sigma b \left(\sigma \left(\frac{\sigma + b + 3}{\sigma - b - 1} \right) - 1 \right).$$

Multiplying both equations through by $(\sigma - b - 1)$ yields a simpler form, and rearranging gives that

$$0 = -(\sigma - b - 1)\omega^3 + 2\omega\sigma b(\sigma + 1)$$
$$0 = -\omega^2(\sigma^2 - (b+1)^2) + 2\sigma b\left(\sigma^2 + \sigma(b+2) + b + 1\right).$$

Solving for ω^2 in both equations then implies that

$$\omega^2 = \frac{2\sigma b(\sigma + 1)}{\sigma - b - 1}$$
$$\begin{aligned} \omega^2 &= \frac{2\sigma b(\sigma^2 + \sigma(b+2) + b + 1)}{\sigma^2 - (b+1)^2} \\ &= 2\sigma b \frac{(\sigma + b + 1)(\sigma + 1)}{(\sigma + b + 1)(\sigma - b - 1)} \\ &= \frac{2\sigma b(\sigma + 1)}{\sigma - b - 1}, \end{aligned}$$

hence both the real part and the imaginary part coincide and are equal to give a pair of eigenvalues! Indeed, they are conjugate to one another:

$$\lambda_{1,2} = \pm i \sqrt{2\sigma b(\sigma + 1)}\sigma - (b + 1).$$

Note that here $\sigma > b + 1$ else ω becomes complex again, which cannot occur since ω must be real for this solution to exist.

(c) The fundamental theorem of algebra states that a third eigenvalue should exist in the characteristc equation as specified in part (a). We begin with factoring out the two eigenvalues previously found of the characteristic equation. We have that

$$\begin{aligned} (\lambda - i\omega)(\lambda + i\omega) &= \lambda^2 - i\lambda\omega + i\lambda\omega - i^2\omega^2 \\ &= \lambda^2 + \omega^2. \end{aligned}$$

Let c denote the third eigenvalue. We then have that

$$0 = (\lambda^2 + \omega^2)(\lambda + c) = \lambda^3 + \lambda^2(b + \sigma + 1) + \lambda b(\sigma + r_H) + 2\sigma b(r_H - 1).$$

Expanding our the left hand side then gives the relation that

$$\lambda^3 + \lambda^2 c + \omega^2 \lambda + \omega^2 c = \lambda^3 + \lambda^2(b + \sigma + 1) + \lambda b(\sigma + r_H) + 2\sigma b(r_H - 1),$$

which suggests that $c = (b + \sigma + 1)$, but this can be shown by subsituting in our values of ω:

$$0 = \lambda^3 + \lambda^2 c + \left(\frac{2\sigma b(\sigma + 1)}{\sigma - b - 1} \right) \lambda + \left(\frac{2\sigma b(\sigma + 1)}{\sigma - b - 1} \right) c$$

$$
\begin{aligned}
&= \lambda^3 + \lambda^2(b+\sigma+1) + \lambda b(\sigma + r_H) + 2\sigma b(r_H - 1) \\
&= \lambda^3 + \lambda^2(b+\sigma+1) + \lambda b\sigma \left(1 + \left(\frac{\sigma+b+3}{\sigma-b-1}\right)\right) + 2\sigma b \left(\sigma \left(\frac{\sigma+b+3}{\sigma-b-1}\right) - 1\right) \\
&= \lambda^3 + \lambda^2(b+\sigma+1) + \left(\frac{2\sigma b(\sigma+1)}{\sigma-b-1}\right)\lambda + \left(\frac{2\sigma b(\sigma+1)}{\sigma-b-1}\right)(b+\sigma+1),
\end{aligned}
$$

which shows that the claim is true. Therefore the three eigenvalues are

$$
\begin{aligned}
\lambda_1 &= +i\sqrt{\frac{2\sigma b(\sigma)}{\sigma-b-1}}, \\
\lambda_2 &= -i\sqrt{\frac{2\sigma b(\sigma)}{\sigma-b-1}}, \\
\lambda_3 &= -(\sigma+b+1),
\end{aligned}
$$

which is what I was looking to determine.

10.3.4

(a) To determine the fixed points of the quadratic map $x_{n+1} = f(x_n) = x_n^2 + c$, one can consider the method of finding the zeroes of the function $y(x) = x^2 - x + c$. This is a relatively easy excercise by the use of the quadratic formula, and one finds that

$$x_{\pm}^* = \frac{1 \pm \sqrt{1-4c}}{2}.$$

The stability of the fixed points are given by the slope of the multiplier $f'(x^*) = 2x^*$. Consider the general equality $f(x) = x$, hence $f'(x) = 1$. Whenever $f'(x^*) < 1$, our fixed point is stable. For $f'(x^*) > 1$, the fixed point is unstable. We have that, as a function of c,

$$\begin{aligned} f'(x_+^*) &= 1 + \sqrt{1-4c} \\ f'(x_-^*) &= 1 - \sqrt{1-4c}, \end{aligned}$$

which implies that x_+^* is always unstable, since $\sqrt{1-4c} > 0$ for $c < \frac{1}{4}$. Similarly, x_-^* is always a stable fixed point, since $-\sqrt{1-4c} < 0$ for $c < \frac{1}{4}$. We obtain a semistable fixed point when $c = \frac{1}{4}$. These roots become complex for $c > \frac{1}{4}$, so we will stick to our analysis of $c \leq \frac{1}{4}$.

(b) As previously noted, we obtain fixed points 'out of the blue' when $c = \frac{1}{4}$, and a stable and unstable point whenever $c < \frac{1}{4}$. This directly implies that we have obtained a saddle-node bifucation at the critical value $c_c = \frac{1}{4}$.

(c) Now consider the possibility of a two-cycle. A two-cycle only occurs if there exists points p and q such that $f(f(q)) = q$ and $f(f(p)) = p$, while $f(q) = p$ and $f(p) = q$. We search to find such points. To begin out analysis, we will proceed by considering the explicit equation of the second iterated map $f(f(x)) = (x^2 + c)^2 + c$. Expanded, we have the polynomial

$$f(f(x)) = x^4 + 2cx^2 + c^2 + c.$$

Like the fixed points of the first iterated system, the fixed points of the second iterated system are equivalently the points p and q which yield the two-cycle coordinates, hence we search for the solutions of the polynomial $x = x^4 + 2cx^2 + c^2 + c$, or the roots of

$$y(x) = x^4 - 2cx^2 - x + c^2 + c.$$

Believe it or not, but our previously determined fixed points $x_{\pm}^* = \frac{1 \pm \sqrt{1-4c}}{2}$ are already solutions of this polynomial:

$$\begin{aligned} y(x_{\pm}^*) &= \left(\frac{1 \pm \sqrt{1-4c}}{2}\right)^4 + 2c\left(\frac{1 \pm \sqrt{1-4c}}{2}\right)^2 - \left(\frac{1 \pm \sqrt{1-4c}}{2}\right) + c^2 + c \\ &= \left(\frac{2 - 4c \pm 2\sqrt{1-4c}}{4}\right)^2 + 2c\left(\frac{2 - 4c \pm 2\sqrt{1-4c}}{4}\right) - \left(\frac{1 \pm \sqrt{1-4c}}{2}\right) + c^2 + c \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{16}\left(8 \pm 8\sqrt{1-4c} + 16c^2 \mp 16c\sqrt{1-4c} - 32c\right) + 2c\left(\frac{2-4c \pm 2\sqrt{1-4c}}{4}\right) - \left(\frac{1 \pm \sqrt{1-4c}}{2}\right) + c^2 + c \\
&= \frac{1}{2} \pm \frac{\sqrt{1-4c}}{2} + c^2 \mp c\sqrt{1-4c} + c - 2c^2 \pm c\sqrt{1-4c} - 2c - \frac{1}{2} \mp \frac{\sqrt{1-4c}}{2} + c^2 + c \\
&= \pm\frac{\sqrt{1-4c}}{2} \mp \frac{\sqrt{1-4c}}{2} \pm c\sqrt{1-4c} \mp c\sqrt{1-4c} + \frac{1}{2} - \frac{1}{2} + 2c^2 - 2c^2 + 2c - 2c \\
&= 0.
\end{aligned}
$$

It now suffices to find the other two zeroes of this equation. Note that

$$(x - x_-^*)(x - x_+^*) = \frac{1}{4}\left(4x^2 - 4x + 4c\right) = x^2 - x + c.$$

Dividing the polynomial $(x^4 + 2cx^2 - x + c^2 + c)$ by the product of the roots found above $x^2 - x + c$, we obtain a quadratic whose roots are the solutions of the points of the two-cylce. Briefly, and easily verifiable, we have that

$$(x^2 - x + c)(x^2 + x + 1 + c) = x^4 + 2cx^2 - x + c^2 + c$$

(I did not want to write out all the steps). Hence the solutions to $x^2 + x + c + 1$ yields the points p and q of the two-cycle. The quadratic formula gives that

$$p,\, q = \frac{-1 \pm \sqrt{1 - 4(1+c)}}{2} = \frac{-1 \pm \sqrt{-3-4c}}{2},$$

hence p and q must only exist when $-3 - 4c > 0$, or when $c < -\frac{3}{4}$. This is the value of c for when the two-cycle occurs. Therefore our roots of the polynomial are

$$x_\pm^* = \frac{1 \pm \sqrt{1-4c}}{2}, \qquad p = \frac{-1 + \sqrt{-3-4c}}{2}, \qquad q = \frac{-1 - \sqrt{-3-4c}}{2}.$$

It is important to note that, when $c < -\frac{3}{4}$ when the two-cycle exists, the fixed points $x_\pm^*$ still remain. Here, we obtain that the two-cycle is superstable, since there still remains a fixed point in the orbit. This two-cycle is always superstable once birthed.

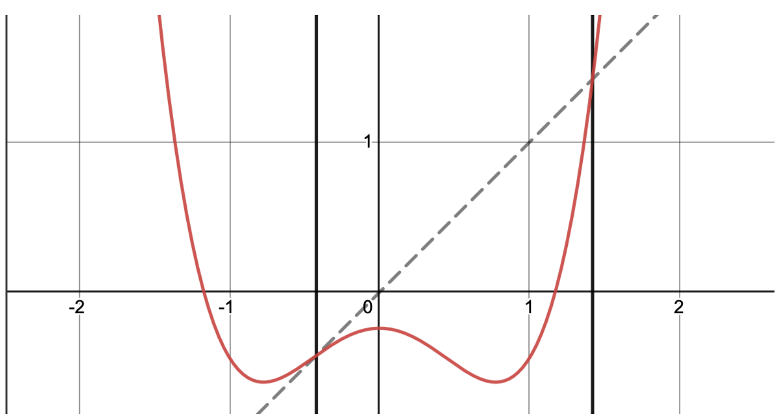


Figure 8: c=-0.6, only fixed points exist.

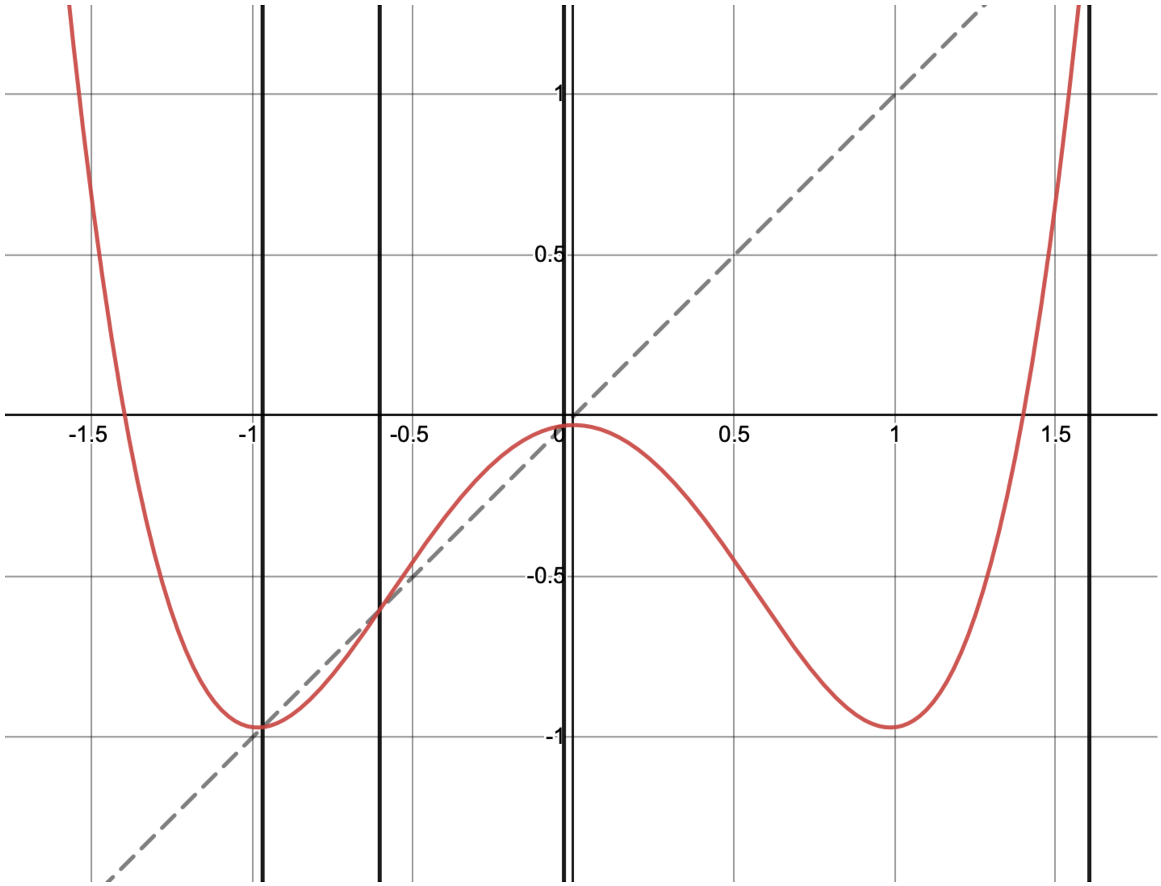


Figure 9: c=-1, bifurcation of two-cycle has occured.

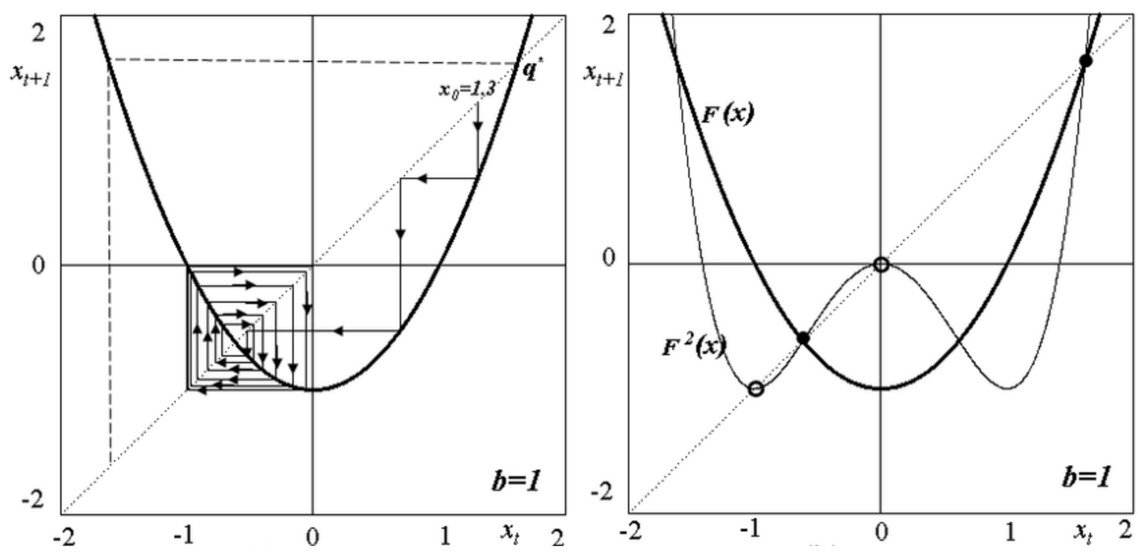


Figure 10: Two-cycle visualization in cobweb plot. Taken from *JOUR - "Introduction to Discrete Nonlinear Dynamical Systems" (Agliari, Anna), (Bischi, Gian-Italo), (Gardini, Laura), (Sushko, Iryna), 2022/12/05. See Introduction to Discrete Nonlinear Systems.*

(d) The corresponding bifurcation diagram is drawn below:

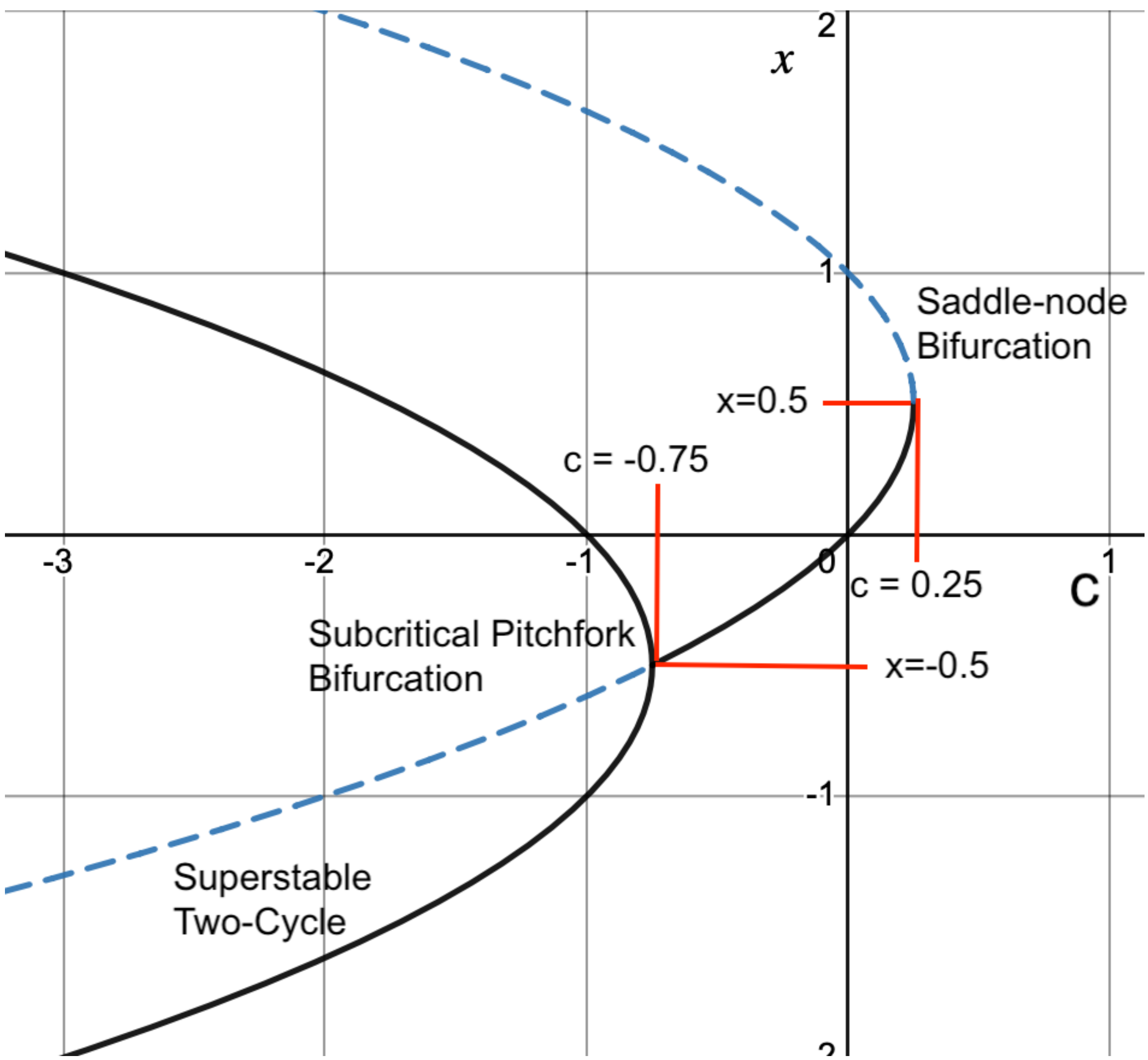


Figure 11: Bifucation diagram of the system. Blue Dashed lines represent unstable fixed points, while black dashed lines represent fixed points and two-cycle orbits.

The fixed points of the system are again

$$x^*_{\pm} = \frac{1 \pm \sqrt{1-4c}}{2}, \qquad p = \frac{-1+\sqrt{-3-4c}}{2}, \qquad q = \frac{-1-\sqrt{-3-4c}}{2},$$

while p and q are only valid for $c \leq -\frac{3}{4}$, and $x^*_{\pm}$ are only valid for $c \leq \frac{1}{4}$.