MAT235 Tutorial 11: Solutions to Additional Problems

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5. Show that $\int_{\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. *This is called the Gaussian integral, I. Hint: extend the integral into two-dimensions, then transform into polar coordinates. We can write*

$$I^2 = \int_{\infty}^{\infty} e^{-x^2}\,dx \int_{\infty}^{\infty} e^{-y^2}\,dy.$$

Solution: Let $I = \int_{\infty}^{\infty} e^{-x^2}\,dx$. This is a single-variable integral, but we can extend the integral into two dimensions by squaring it:

$$I^2 = \int_{\infty}^{\infty} e^{-x^2}\,dx \int_{\infty}^{\infty} e^{-y^2}\,dy \tag{1}$$

Since the bounds of integration are variable-independent and are over all of two-dimensional orthogonal space ($\mathbb{R}^2$), we can write

$$I^2 = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dxdy \tag{2}$$

where the bounds of integration extend to infinity along each axis. Changing the integral into polar coordinates allows us to complete the integral by letting $r^2 = x^2 + y^2$, so $dxdy \to rd\theta dr$ (the bounds of integration are still over all space):

$$\begin{aligned}
I^2 &= \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dxdy & (3)\\
&= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,rdrd\theta & (4)\\
&= \int_0^{2\pi} d\theta \int_0^{\infty} re^{-r^2}\,dr & (5)\\
&= (2\pi)\left(-\frac{1}{2}\lim_{r\to\infty}[e^{-r^2} - e^{(0)}]\right) & (6)\\
&= \pi & (7)
\end{aligned}$$

Therefore $I^2 = \pi$, so $I = \pm\sqrt{\pi}$. However, since the Gaussian function is strictly positive $e^{-x^2} > 0$, the area of the integral must also be positive for any value of x, so we choose the positive value of I. Hence $I = \sqrt{\pi}$.

6. The plane $x - y + z = 2$ intersects the cylinder $x^2 + y^2 = 4$ in an ellipse. Find the points on the ellipse closest to and furthest from the origin. *Hint: to optimize distance* $\sqrt{x^2 + y^2 + z^2}$ *in* $\mathbb{R}^3$*, we can optimize* $f(x, y, z) = x^2 + y^2 + z^2$ *as it's derivative is simpler.*

   Solution: To optimize distance in $\mathbb{R}^3$, we optimize $\sqrt{x^2 + y^2 + z^2}$ as given by the Pythagorean theorem. Suppose we have two distances a, b which are non-negative with $a < b$. Multiplying both sides by a and b independently give that $a^2 < ab$ and $ab < b^2$. Thus $a^2 < ab < b^2$, so $a^2 < b^2$. Therefore, to simplify the derivatives of the problem, we can optimize the square of the distance, the function $f(x, y, z) = x^2 + y^2 + z^2$ whose level surfaces are spheres in $\mathbb{R}^3$. This is a two constraint problem, since the gradient of the intersection (the ellipse) is <u>not</u> equal to the gradient of the plane and the cylinder together. Therefore we use

$$\nabla f = \lambda \nabla g + \mu \nabla h \tag{1}$$
$$g(x, y, z) = c \tag{2}$$
$$h(x, y, z) = k \tag{3}$$

   to determine a system of equations whose constrains imply an optimization with respect to the intersection of both surfaces. Let $g(x, y, z) = x - y + z$ be the plane and $h(x, y, z) = x^2 + y^2$ be the cylinder along the z-axis. The gradients are

$$\nabla f(x, y, z) = \langle 2x, 2y, 2z \rangle \tag{4}$$
$$\nabla g(x, y, z) = \langle 1, -1, 1 \rangle \tag{5}$$
$$\nabla h(x, y, z) = \langle 2x, 2y, 0 \rangle. \tag{6}$$

   Matching the x, y, z components respectively give a 5-equation system from (1), (2), (3) above:

$$2x = \lambda + 2\mu x \tag{7}$$
$$2y = -\lambda + 2\mu y \tag{8}$$
$$2z = \lambda \tag{9}$$
$$x - y + z = 2 \tag{10}$$
$$x^2 + y^2 = 4. \tag{11}$$

   Let's begin by examining the implications of equations (7) and (8). Adding these equations imply that

$$x + y = \mu(x + y) \tag{12}$$

   which has solutions $\mu = 1$ or $x = -y$. If $\mu = 1$, then $\lambda = z = 0$ (since $2x - 2(1)x = \lambda$), which simplifies the constraints to

$$x - y = 2 \tag{13}$$
$$x^2 + y^2 = 4. \tag{14}$$

   (13) implies $x = 2 + y$, so $y^2 + (2 + y)^2 = 4 \implies 2y(y + 2) = 0$ so $y = 0, -2$. This corresponds to the points $(2, 0, 0)$ and $(0, -2, 0)$ from equation (13). Both of these points have a distance 2 from the origin. If $x = -y$ to satisfy equation (12), then the constraints become

$$2x + z = 2 \tag{15}$$
$$2x^2 = 4. \tag{16}$$

Again, (16) implies that $x = \pm\sqrt{2}$, so $y = \mp\sqrt{2}$, and $z = 2 \mp 2\sqrt{2}$. The next set of points we examine are then $\left(\sqrt{2}, -\sqrt{2}, 2 - 2\sqrt{2}\right)$ and $\left(-\sqrt{2}, \sqrt{2}, 2 + 2\sqrt{2}\right)$. The distance of these points are

$$\sqrt{2 + 2 + (2 \mp 2\sqrt{2})^2} = \sqrt{16 \pm 8\sqrt{2}} \tag{17}$$

which are both greater than 2, since $\sqrt{16} = 4 > 2$. Therefore the closest set of points to the origin in optimizing distance are $(2, 0, 0)$ and $(0, -2, 0)$.

7. Let $H = \{x^2 + y^2 + z^2 \leq 4,\, z \geq 0\}$ be the upper solid hemisphere of radius 2. Express the volume of H as an iterated double integral with integration order $dydx$.

Solution: We begin by identifying the integrand. In between the upper hemisphere of radius 2 and the xy-plane, we have z, so we integrate z between 0 and $z = +\sqrt{4 - x^2 - y^2}$ (upper hemisphere is +):

$$\int_0^{\sqrt{4-x^2-y^2}} dz = \sqrt{4 - x^2 - y^2}. \tag{1}$$

We now determine the integration bounds Since we are integrating only in x and y, the integrand must be a function of x and y only and this is what is given above by the z-integration in equation (1). The bounds of x and y are given when $z = 0$, the circle of radius 2 about the origin in the xy plane.

y must then integrate as a function of x, since we integrate y first, and these bounds are given by $y = \pm\sqrt{4 - x^2}$ to satisfy the $z = 0$ condition. We lastly observe that x must tend from $-2 \to 2$ to satisfy the $z = 0$ and $y = 0$ conditions. We therefore have

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4 - x^2 - y^2}\, dydx \tag{2}$$

as an iterated double integral with integration order $dydx$.

8. Determine the quadratic (2nd order) two-dimensional Taylor polynomial $Q(x,y)$ of the function $f:\mathbb{R}^2\to\mathbb{R}$ given by $f(x,y)=\arctan(x+2y)$ centered about the point (1,0).

Solution: A second-order quadratic Taylor polynomial about a point $\mathbf{p}=(a,b)$ is given by

$$P_2(\mathbf{x}) = f(\mathbf{p}) + \nabla f\cdot(\mathbf{x}-\mathbf{p}) + \frac{1}{2}(\mathbf{x}-\mathbf{p})^T Hf(\mathbf{p})(\mathbf{x}-\mathbf{p}) \tag{1}$$

where $\mathbf{x}\equiv(x,y)$ and $Hf(\mathbf{p})=\begin{pmatrix} f_{xx}(\mathbf{p}) & f_{xy}(\mathbf{p}) \\ f_{yx}(\mathbf{p}) & f_{yy}(\mathbf{p})\end{pmatrix}$ is the Hessian matrix (if you haven't seen this before, it's ok - just compare your solution with mine).

For the function $f(x,y)=\arctan(x+2y)$, we begin by calculating all partial derivatives up to second order.

$$f_x(x,y) = \frac{1}{1+(x+2y)^2} \tag{2}$$

$$f_y(x,y) = \frac{1}{1+(x+2y)^2}\cdot 2 \tag{3}$$

$$f_{xx}(x,y) = -\frac{1}{1+(x+2y)^2}\cdot 2(x+2y) \tag{4}$$

$$f_{yy}(x,y) = -\frac{2}{1+(x+2y)^2}\cdot 4(x+2y) \tag{5}$$

$$f_{xy}(x,y) = -\frac{2}{1+(x+2y)^2}\cdot 2(x+2y) = f_{yx}(x,y) \tag{6}$$

where (6) follows from Clairaut's theorem. We evaluate these derivatives at the point $\mathbf{p}=(1,0)$, which give

$$f_x(1,0) = \frac{1}{2} \tag{7}$$

$$f_y(1,0) = 1 \tag{8}$$

$$f_{xx}(1,0) = -1 \tag{9}$$

$$f_{yy}(1,0) = -4 \tag{10}$$

$$f_{xy}(1,0) = -2 = f_{yx}(1,0). \tag{11}$$

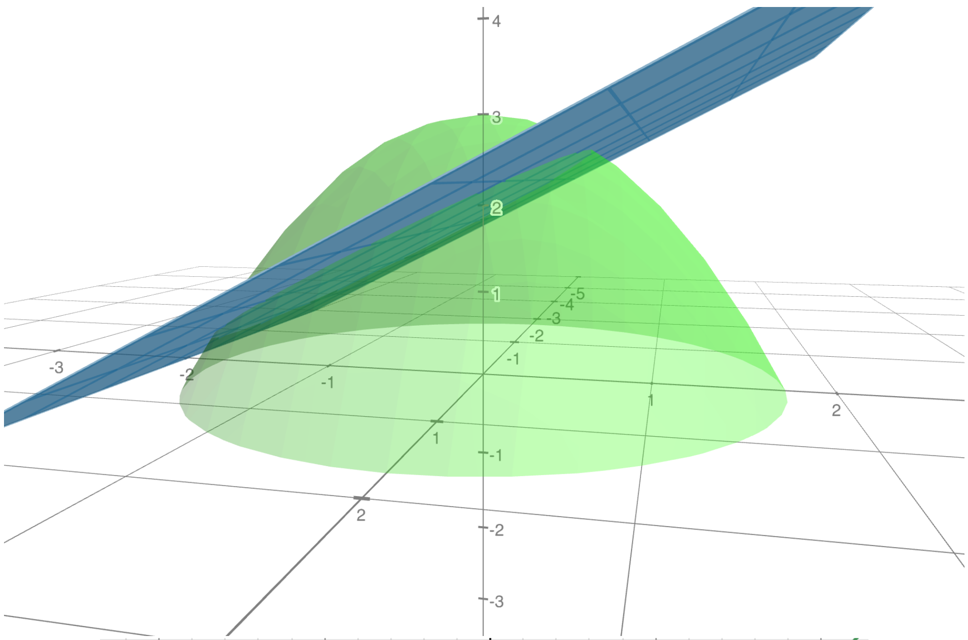
We then work to simplify equation (1):

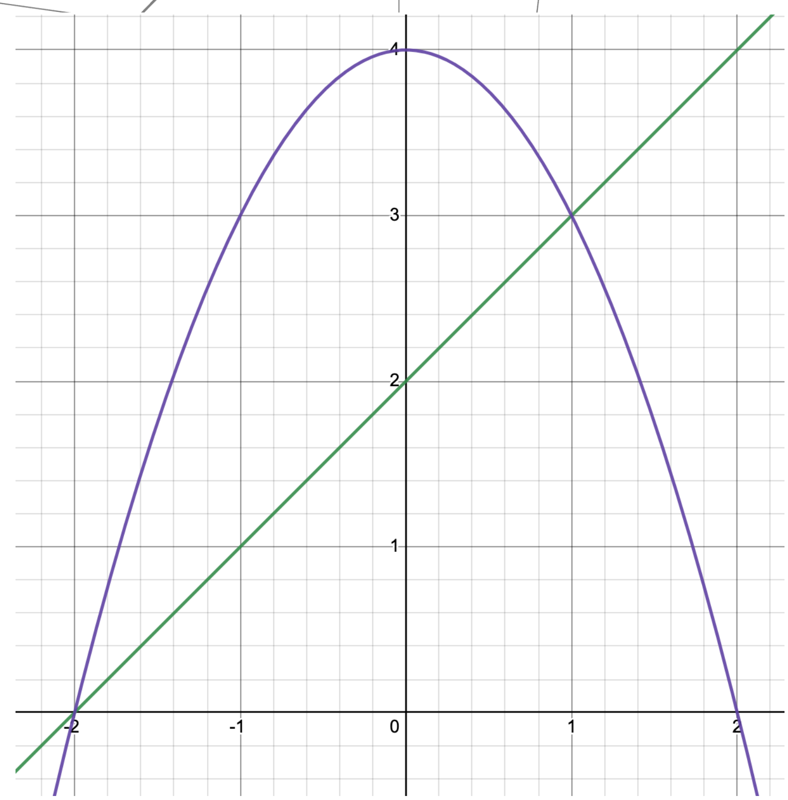
$$\begin{aligned}
P_2(x,y) &= \frac{1}{2} + \left(\frac{1}{2},-1\right)\begin{pmatrix} x-1 \\ y\end{pmatrix} + \frac{1}{2}(x-1,y)\begin{pmatrix} -1 & -2 \\ -2 & -4\end{pmatrix}\begin{pmatrix} x-1 \\ y\end{pmatrix} && (12)\\
&= \frac{\pi}{4} + \frac{1}{2}(x-1) + y - \frac{1}{2}(x-1,y)\begin{pmatrix} (x-1)+2y \\ 2(x-1)+4y\end{pmatrix} && (13)\\
&= \frac{\pi}{4} + \frac{1}{2}(x-1) + y - \frac{1}{2}\left((x-1)^2 + 2y(x-1) + 2y(x-1) + 4y^2\right) && (14)\\
&= \frac{\pi}{4} + \frac{1}{2}(x-1) + y - \frac{1}{2}(x-1)^2 + 2y(x-1) + 2y^2 && (14)\\
&= \frac{\pi}{4} + \frac{1}{2}x - \frac{1}{2} + y - \frac{1}{2}x^2 - \frac{1}{2} + x + 2yx - 2y + 2y^2 && (15)\\
&= \frac{\pi}{4} - 1 + \frac{3}{2}x + 3y - \frac{1}{2}x^2 - 2xy - 2y^2 && (16)
\end{aligned}$$

which is the second order Taylor polynomial of $f(x,y)$ about the point $\mathbf{p}=(1,0)$.

9. Consider the solid S above $z = 0$, below $z = y + 2$, and below the paraboloid $z = 4 - x^2 - y^2$. Use <u>slices</u> to construct the volume of S in terms of iterated integrals with the order $dxdydz$. Explain the geometry of your process. Do not evaluate the integral (it's not fun).

Solution: The first suggested step would be to draw a diagram of the shape.





Above are attached images of the full volume and a slice at $x = 0$. At $x = 0$, we observe that the paraboloid and the plane intersect at two locations:

$$y + 2 = 4 - y^2 \implies y^2 + y - 2 = (y+2)(y-1) = 0 \tag{1}$$

which is $y = -2, y = 1$.

We begin by identifying the bounds of the integration.

Since we are integrating x first, we can find that $z = 4 - x^2 - y^2$ implies that x must range from $-\sqrt{4 - y^2 - z} \to \sqrt{4 - y^2 - z}$. It may appear confusing how the $y + 2$ plane is incorporated in this, and the simple answer is, it's not. This is because the plane is independent of x, only a function of y (and z). Therefore when we integrate x, we obtain functions of y and z from which we can impose the bounds on.

The next observation would to notice that y must range from -2 to 2, but under two different conditions. The first condition is when $-2 \leq y \leq 1$, the plane bounds the volume. The second is $1 \leq y \leq 2$, when the paraboloid bounds the volume. We can then integrate $y = z - 2 \to 1$, when the plane bounds, then add the integral for when $y = 1 \to \sqrt{4 - z}$ (remember that we've already integrated x, which is now a function of y and z). This produces a function of only z.

Lastly, the bounds for z are above the xy plane (so $z \geq 0$) and always less than 3, which is the most maximal point of intersection of the surfaces: $(1, 4 - (1)^2) = (1, 3)$. The integral in x is

$$g(y, z) = \int_{-\sqrt{4-y^2-z}}^{\sqrt{4-y^2-z}} dx \tag{2}$$

and in y,

$$h(z) = \int_{z-2}^{1} g(y, z)\, dy + \int_{1}^{\sqrt{4-z}} g(y, z)\, dy \tag{3}$$

and lastly in z,

$$\int_0^3 h(z)\, dz. \tag{4}$$

Putting (2), (3) and (4) altogether gives the volume inside the solid:

$$V = \int_0^3 \left[\int_{z-2}^{1} \int_{-\sqrt{4-y^2-z}}^{\sqrt{4-y^2-z}} dxdy + \int_{1}^{\sqrt{4-z}} \int_{-\sqrt{4-y^2-z}}^{\sqrt{4-y^2-z}} dxdy \right] dz \tag{5}$$

10. Consider the particle in an infinitely deep potential well of width L along $[0, L]$. The time-dependent wavefunction is given by $\psi(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{iE_n t}{\hbar}}$ where the c_n are coefficients, $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$ is the energy of the particle, the $\sqrt{\frac{2}{L}}$ yields probabilistic normalization, and m is the mass of the particle. Show that $\psi(x,t)$ satisfies the Schrödinger equation on the domain $0 \leq x \leq L$,

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t).$$

Solution: We begin by mentioning the linearity of the derivative operator on functions. For two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ and a real constant a, then $\frac{d}{dx}(f + ag) = \frac{df}{dx} + a\frac{dg}{dx}$. This single variable definition extends to multivariable partial derivatives as well.

We show that $\psi(x,t)$ satisfies the Schrödinger equation by calculating the partial derivatives in t and x. We can seperate ψ into positional and temporal components, $\psi(x,t) = \sum_{n=1}^{\infty} \varphi_n(x) e^{-\frac{iE_n t}{\hbar}}$, where the $\varphi_n(x)$ are 'stationary states' given by $\varphi_n(x) = c_n\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$. We find that

$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial t}\left(\sum_{n=1}^{\infty} \varphi_n(x)\, e^{-\frac{iE_n t}{\hbar}}\right) \tag{1}$$

$$= \sum_{n=1}^{\infty} \varphi_n(x) \left(\frac{\partial}{\partial t} e^{-\frac{iE_n t}{\hbar}}\right) \tag{2}$$

$$= \sum_{n=1}^{\infty} \varphi_n(x)\, e^{-\frac{iE_n t}{\hbar}} \cdot \frac{-iE_n}{\hbar}. \tag{3}$$

Similarly,

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial t}\left(\sum_{n=1}^{\infty} \varphi_n(x)\, e^{-\frac{iE_n t}{\hbar}}\right) \tag{4}$$

$$= \sum_{n=1}^{\infty} c_n\sqrt{\frac{2}{L}} \left(\frac{\partial^2}{\partial x^2} \sin\left(\frac{n\pi x}{L}\right)\right) \tag{5}$$

$$= \sum_{n=1}^{\infty} c_n\sqrt{\frac{2}{L}} \left(-\frac{n^2\pi^2}{L^2} \sin\left(\frac{n\pi x}{L}\right)\right) \tag{6}$$

$$= -\sum_{n=1}^{\infty} \frac{n^2\pi^2}{L^2}\varphi_n(x) e^{-\frac{iE_n t}{\hbar}}. \tag{7}$$

Plugging (3) and (6) into the Schrödinger equation yields

$$i\hbar\left(\sum_{i=1}^{\infty} \varphi_n(x)\, e^{-\frac{iE_n t}{\hbar}} \cdot \frac{-iE_n}{\hbar}\right) = -\frac{\hbar^2}{2m}\left(-\sum_{n=1}^{\infty} \frac{n^2\pi^2}{L^2}\varphi_n(x) e^{-\frac{iE_n t}{\hbar}}\right) \tag{8}$$

which must be true for every n. After distributing the factors of i and $\hbar$ and recalling that $i^2 = -1$, we substitute $E_n = \dfrac{n^2\pi^2\hbar^2}{2mL^2}$ and find that

$$\sum_{n=1}^{\infty} \varphi_n(x) e^{-\frac{iE_nt}{\hbar}} E_n = \sum_{n=1}^{\infty} \frac{\hbar^2}{2m} \cdot \frac{n^2\pi^2}{L^2} \varphi_n(x) e^{-\frac{iE_nt}{\hbar}} \tag{9}$$

$$\implies \sum_{n=1}^{\infty} \varphi_n(x) e^{-\frac{iE_nt}{\hbar}} E_n = \sum_{n=1}^{\infty} \varphi_n(x) e^{-\frac{iE_nt}{\hbar}} E_n \tag{10}$$

$$\implies \psi(x,t) = \psi(x,t) \tag{11}$$

hence $\psi(x,t)$ satisfied the Schrödinger equation.