MAT235 Tutorial 11

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Tutorial this week will focus on solving multi-dimensional integrals and preparing for Term Test 3 (February 6). There is no quiz this week. Problems 1-4 will be taken up in class, while all additional problems are for TT3 review and won't be taken up during tutorial time.

1. Determine the volume of the solid bounded between the cone $z = \sqrt{x^2 + y^2}$ and the unit sphere $x^2 + y^2 + z^2 = 1$. *Hint: use polar coordinates.*

2. Write the following sum of triple integrals as one iterated volume integral in spherical coordinates. For extra practice, show that the value is $\dfrac{15\pi}{16}$.

$$\int_0^{\pi/2} \int_0^1 \int_{\sqrt{1-r^2}}^{\sqrt{4-r^2}} r^2 \cos\theta \, dz dr d\theta + \int_0^{\pi/2} \int_1^2 \int_0^{\sqrt{4-r^2}} r^2 \cos\theta \, dz dr d\theta$$

3. Let E be the solid shell above the xy-plane bounded by the concentric spheres of radius a and b, where $0 < a < b$. Determine an expression for the volume of E in terms of a and b.

4. Determine the volume of the solid V which lies between the paraboloid $z = 24 - x^2 - y^2$ and the cone $z = 2\sqrt{x^2 + y^2}$.

Additional Problems:

5. Show that $\displaystyle\int_{\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$. *This is called the Gaussian integral, I. Hint: extend the integral into two-dimensions, then transform into polar coordinates. We can write*

$$I^2 = \int_{\infty}^{\infty} e^{-x^2}\, dx \int_{\infty}^{\infty} e^{-y^2}\, dy.$$

6. The plane $x - y + z = 2$ intersects the cylinder $x^2 + y^2 = 4$ in an ellipse. Find the points on the ellipse closest to and furthest from the origin. *Hint: to optimize distance $\sqrt{x^2 + y^2 + z^2}$ in $\mathbb{R}^3$, we can optimize $f(x, y, z) = x^2 + y^2 + z^2$ as it's derivative is simpler.*

7. Let $H = \{x^2 + y^2 + z^2 \leq 4,\ z \geq 0\}$ be the upper solid hemisphere of radius 2. Express the volume of H as an iterated <u>double</u> integral with integration order $dydx$.

8. Determine the quadratic (2nd order) two-dimensional Taylor polynomial $Q(x, y)$ of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = \arctan(x + 2y)$ centered about the point (1,0).

9. Consider the solid S above $z = 0$, below $z = y + 2$, and below the paraboloid $z = 4 - x^2 - y^2$. Use <u>slices</u> to construct the volume of S in terms of iterated integrals with the order $dxdydz$. Explain the geometry of your process. Do not evaluate the integral (it's not fun).

10. Consider the particle in an infinitely deep potential well of width L along $[0, L]$. The time-dependent wavefunction is given by $\psi(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{iE_n t}{\hbar}}$ where the c_n are coefficients, $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$ is the energy of the particle, the $\sqrt{\frac{2}{L}}$ yields probabilistic normalization, and m is the mass of the particle. Show that $\psi(x,t)$ satisfies the Schrödinger equation on the domain $0 \leq x \leq L$,

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t).$$