**MAT235 Tutorial 14: Solutions to Additional Problems**

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5. Consider the parallelogram $P = \{(x,y) \in \mathbb{R}^2 : 0 \leq x - y \leq 3, 1 \leq x + y \leq 3\}$ and its mass-density function $\delta : P \to \mathbb{R}$ given by $\delta(x,y) = (x+y)^2$. Use a change of variables to find the mass of $P$.

Solution: We can choose a change of variables to transform the parallelogram into a rectangle of side lengths 3 and 2. Let $u = x - y$ and $v = x + y$, so that $P$ becomes

$$P = \{(u,v) \in \mathbb{R}^2 : 0 \leq u \leq 3, 1 \leq v \leq 3\}. \tag{1}$$

Under this substitution the mass density function also becomes $\delta(u,v) = v^2$. The mass of $P$ is given by integration over the volume of the mass density function, $\text{mass}(P) = \iint_P \delta(u,v)\, dA$. The bounds of the integration are given by (1), but to determine the area element we must compute the stretch factor goverened by the Jacobian. Solving for $u$ and $v$ gives that $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(v-u)$, so we find

$$\begin{aligned}
dA &= |\det D(x,y)|\, dudv && (2)\\
&= \det \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} dudv && (3)\\
&= \det \begin{pmatrix} \dfrac{1}{2}\dfrac{\partial}{\partial u}(u+v) & \dfrac{1}{2}\dfrac{\partial}{\partial v}(u+v) \\[2ex] \dfrac{1}{2}\dfrac{\partial}{\partial u}(v-u) & \dfrac{1}{2}\dfrac{\partial}{\partial v}(v-u) \end{pmatrix} dudv && (4)\\
&= \det \begin{pmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\[2ex] -\dfrac{1}{2} & \dfrac{1}{2} \end{pmatrix} dudv && (5)\\
&= \left(\frac{1}{4} + \frac{1}{4}\right) dudv && (6)\\
&= \frac{1}{2}\, dudv.
\end{aligned}$$

Thus the stretch factor is $\frac{1}{2}$. The mass is then

$$\begin{aligned}
\text{mass}(P) &= \iint_P \delta(u,v)\, dA && (7)\\
&= \int_1^3 \int_0^3 v^2 \, \frac{1}{2} dudv && (8)\\
&= \frac{1}{2} \int_0^3 du \int_1^3 v^2\, dv && (9)
\end{aligned}$$

$$= \frac{1}{2} \cdot 3 \cdot \frac{1}{3} v^3 \Big|_1^3 \tag{10}$$

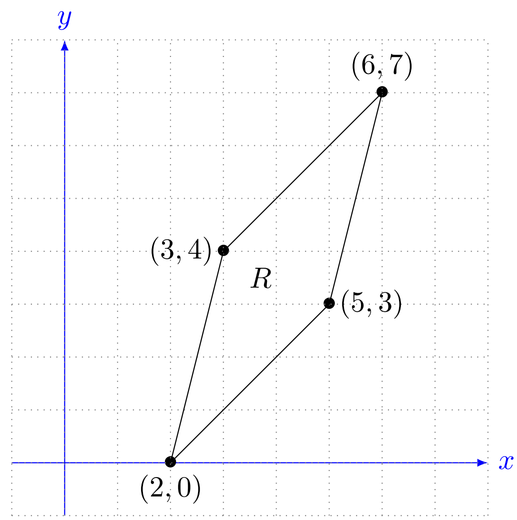
$$= \frac{1}{2} [27 - 1] \tag{11}$$

$$= 13. \tag{14}$$

6. Evaluate $\iint_R (2x - \pi y)\, dA$ where $R$ is the parallelogram with vertices $(2,0),(5,3),(6,7),(3,4)$. *Hint: derive a change of variables.*

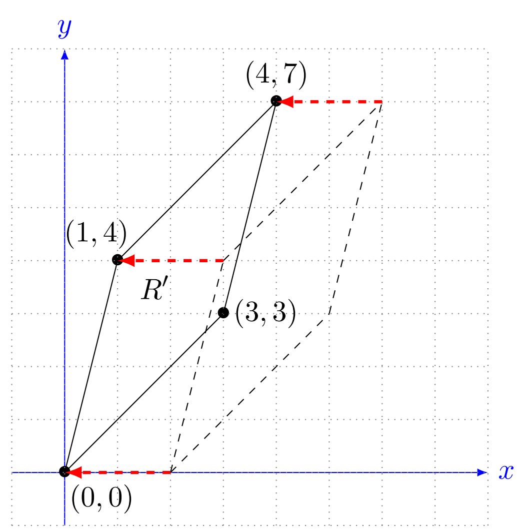
*You are not expected to know linear algebra for MAT235, but I wanted to showcase this problem as it's important to understand how change of variables work in parallel to linear transformations. This problem is on the tougher side, as it requires some thinking and prerequisite knowledge from a course like MAT223.*

Solution: The parallelogram $R$ is shown below:




Let's label the vectors of the parallelogram $\mathbf{a} = (3,4)$ and $\mathbf{b} = (5,3)$ so that the area spanned by this parallelogram is $\mathbf{a} \times \mathbf{b}$. The goal is to determine some kind of linear transformation matrix which takes in vectors in $R$ and maps them to the unit square spanned by $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, that is, $(1,0)$ and $(0,1)$.

We note that $R$ is not centered at the origin, but at (2,0). Thus we can begin by translating $R$ by having every $x \to x-2$. This implies the first change of variables, $x' = x-2$, $y' = y$ (the prime doesn't represent derivative, it just means that this is a secondary coordinate system), implying that $R$ becomes



Since translations don't change the area or shape of $R$ itself, the determinant of the Jacobian remains 1 so $dA' = dA$. The bounds of the integral are still unknown (which is ok!), but the integrand substitutes $x = x' + 2$, hence $2(x' + 2) - \pi y'$ is the new integrand. We have

$$\iint_{R'} (2(x' + 2) - \pi y')\, dA' \tag{1}$$

after the first translation.

Now that $R'$ is fixed at the origin, we can find a linear transformation which takes in vectors $(1, 0)$ and $(0, 1)$ and maps them to $(3, 3)$ and $(1, 4)$ respectively (this is called a transformation matrix, https://en.wikipedia.org/wiki/Transformation_matrix). One can verify that the following matrix multiplication gives that

$$\begin{pmatrix} 3 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} 3 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \tag{3}$$

hence $M = \begin{pmatrix} 3 & 1 \\ 3 & 4 \end{pmatrix}$ is the matrix of the linear transformation: the matrix which takes in vectors in the unit square and maps them to $R'$. However, we want to find a linear transformation which takes vectors in $R'$ and maps them to the unit square, which is equivalently the inverse of $M$. The inverse of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$A^{-1} = \frac{1}{\det A} \mathrm{Adj}(A) \tag{4}$$

$$= \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{5}$$

which, for $M$, is

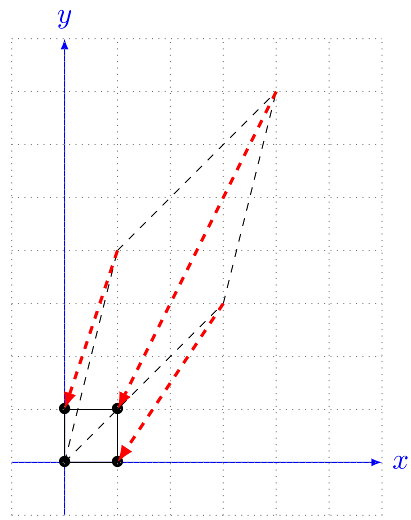
$$M^{-1} = \frac{1}{(3)(4) - (3)(1)} \begin{pmatrix} 4 & -1 \\ -3 & 3 \end{pmatrix} \tag{6}$$

$$= \frac{1}{9} \begin{pmatrix} 4 & -1 \\ -3 & 3 \end{pmatrix} \tag{7}$$

$$= \begin{pmatrix} 4/9 & -1/9 \\ -1/3 & 1/3 \end{pmatrix} \tag{8}$$

Therefore $M^{-1}$ takes vectors in $R'$ and maps them to the unit square, as desired.

How does this relate back to multivariable change of variables? Recall that the determinant of the Jacobian of a transformation $\det D(x, y)$ governs the scaling of unit area under the transformation. Equivalently, in linear algebra, $\det M^{-1}$ governs the stretching of unit area under a linear transformation to take $R'$ to $[0, 1] \times [0, 1]$. Hence $M^{-1}$ is the Jacobian of the transformation which takes points in $R'$ to the unit square.




We therefore have

$$\begin{pmatrix} \dfrac{\partial u}{\partial x'} & \dfrac{\partial u}{\partial y'} \\ \\ \dfrac{\partial v}{\partial x'} & \dfrac{\partial v}{\partial y'} \end{pmatrix} = \begin{pmatrix} 4/9 & -1/9 \\ -1/3 & 1/3 \end{pmatrix} = M^{-1} \tag{9}$$

from which implies that, upon matching components and integrating,

$$u = \frac{1}{9}(4x' - y') \tag{10}$$

$$v = \frac{1}{3}(-x' + y') \tag{11}$$

which is the change of variables which takes in points $(x', y') \in R'$ and maps them to the unit square. One may verify that $R'$ is the set

$$R' = \{(x', y') \in \mathbb{R}^2 \,:\, 0 \le x' - y' \le 3,\, 0 \le 4x' - y' \le 9\}. \tag{12}$$

In order to compute the initial integral, we must determine the bounds of the integral, the stretching factor, and the integrand. It is easy to see that, after our transformation to $(u, v)$ coordinates, that $u$ and $v$ are both bound between 0 and 1 (it is the unit square), thus these are the integration bounds. The area element is

$$dA' = |\det M^{-1}|\, dudv \tag{13}$$

$$= \left| \frac{4}{9}\frac{1}{3} - \left(-\frac{1}{3}\right)\left(-\frac{1}{9}\right) \right| dudv \tag{14}$$

$$= \left| \frac{4-1}{27} \right| dudv \tag{15}$$

$$= \frac{1}{9}\, dudv. \tag{16}$$

The integrand can be determined by solving for $x'$ and $y'$, from which are implicitly given in (10) and (11). We find

$$9u = 4x' - y' \implies y' = 4x' - 9u \tag{17}$$

$$3v = -x' + y' \implies 3v + x' = 4x' - 9u \implies x' = v + 3u \tag{18}$$
$$\implies y' = 4v + 3u \tag{19}$$

thus the integrand becomes

$$2x - \pi y = 2(x' + 2) - \pi y' \tag{20}$$
$$= 2(v + 3u) + 4 - \pi(4v + 3u) \tag{21}$$
$$= 2v - 4\pi v + 6u - 3\pi u + 4 \tag{22}$$
$$= (2 - 4\pi)v + (6 - 3\pi)u + 4 \tag{23}$$

Lastly, combining the first translation, the determined bounds, equation (16) and (23), we can compute the integral:

$$\iint_R (2x - \pi y)\, dA = \int_0^1 \int_0^1 \left[(2 - 4\pi)v + (6 - 3\pi)u + 4\right] \frac{1}{9}\, dudv \tag{24}$$
$$= \frac{1}{9} \int_0^1 \left[(2 - 4\pi)uv + \frac{1}{2}(6 - 3\pi)u^2 + 4u\right] \Big|_0^1 dv \tag{25}$$
$$= \frac{1}{9} \int_0^1 \left[(2 - 4\pi)v + \frac{1}{2}(6 - 3\pi) + 4\right] dv \tag{26}$$
$$= \frac{1}{9} \left[\frac{1}{2}(2 - 4\pi)v^2 + \frac{1}{2}(6 - 3\pi)v + 4v\right] \Big|_0^1 \tag{27}$$
$$= \frac{1}{9} \left[\frac{1}{2}(2 - 4\pi) + \frac{1}{2}(6 - 3\pi) + 4\right] \tag{28}$$
$$= \frac{8}{9} - \frac{7\pi}{18} \tag{29}$$
$$\approx -0.333. \tag{30}$$

7. The parametrization of a hypocycloid is given by

$$x(t) = (a-b)\cos t + b\cos\left(\frac{a-b}{b}\right)t$$
$$y(t) = (a-b)\sin t - b\sin\left(\frac{a-b}{b}\right)t$$

Let $C_H$ be the 5-point hypocycloid curve given by parameters $a = 10$ and $b = 2$, and $0 \le t \le 2\pi$. Let $\mathbf{G} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field $\mathbf{G}(x, y) = \langle x, 2y\rangle$. Compute $\oint_{C_H} \mathbf{G} \cdot d\mathbf{r}$.

Solution: We use the definition of the line integral,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(\mathbf{t})) \cdot \mathbf{r}'(t)\, dt. \tag{1}$$

The bounds of our integral are the bounds of the parametrization input $t$, $[0, 2\pi]$. The parametrization is given

$$\mathbf{r}(t) = \begin{bmatrix} 8\cos t + 2\cos(4t) \\ 8\sin t - 2\sin(4t) \end{bmatrix} \tag{2}$$

from which

$$\mathbf{r}'(t) = \begin{bmatrix} -8\sin t - 8\sin(4t) \\ 8\cos t - 8\cos(4t) \end{bmatrix}. \tag{3}$$

With $\mathbf{G}(x, y) = \langle x, 2y\rangle$, we compute the line integral:

$$\oint_{C_H} \mathbf{G} \cdot d\mathbf{r} = \int_0^{2\pi} \begin{bmatrix} 8\cos t + 2\cos(4t) \\ 16\sin t - 4\sin(4t) \end{bmatrix} \cdot \begin{bmatrix} -8\sin t - 8\sin(4t) \\ 8\cos t - 8\cos(4t) \end{bmatrix} dt \tag{4}$$

$$\begin{aligned} &= \int_0^{2\pi} \big[(8\cos t + 2\cos(4t))(-8\sin t - 8\sin(4t)) \\ &\qquad + (16\sin t - 4\sin(4t))(8\cos t - 8\cos(4t))\big]\, dt \end{aligned} \tag{5}$$

$$\begin{aligned} &= \int_0^{2\pi} \big[-64\cos t\sin t - 16\sin(4t)\cos(4t) - 64\cos t\sin(4t) - 16\sin t\cos(4t) \\ &\qquad + 128\sin t\cos t + 32\sin(4t)\cos(4t) - 128\sin t\cos(4t) - 32\cos t\sin(4t)\big]\, dt \\ &= 0 \end{aligned} \tag{6}$$

since any product of sine and cosine function integrated over its period ($2\pi$) is zero.

*Proof:* Consider the integral $\int_0^{2\pi} \sin(nx)\cos(px)\, dx$ for any integers $n$ and $p$. Using complex exponentials $e^{ix} = \cos x + i\sin x$, we can write

$$\sin(nx) = \frac{1}{2i}\left[e^{inx} - e^{-inx}\right] \tag{7}$$
$$\cos(px) = \frac{1}{2}\left[e^{ipx} + e^{-ipx}\right] \tag{8}$$

(there is no easier way to show this, unless you want to use binomial expansions for $n$-angle arguments) so

$$\int_0^{2\pi} \sin(nx)\cos(px)\, dx = \frac{1}{4i}\int_0^{2\pi} (e^{inx} - e^{-inx})(e^{ipx} + e^{-ipx})\, dx \tag{9}$$

$$
\begin{aligned}
&= \frac{1}{4i}\int_0^{2\pi}\left[e^{ix(n+p)} + e^{ix(n-p)} - e^{ix(-n+p)} - e^{ix(-n-p)}\right] dx && (10)\\
&= \frac{1}{4i}\int_0^{2\pi}\left[e^{ix(n+p)} - e^{-ix(n+p)} + e^{ix(n-p)} - e^{-ix(n-p)}\right] dx && (11)\\
&= \frac{1}{4i}\int_0^{2\pi}\left[2i\sin((n+p)x) + 2i\sin((n-p)x)\right] dx && (12)\\
&= \frac{1}{2}\int_0^{2\pi}\left[\sin((n+p)x) + \sin((n-p)x)\right] dx && (13)\\
&= \frac{1}{2}\left[-\frac{1}{n+p}\cos((n+p)x) - \frac{1}{n-p}\cos((n-p)x)\right]\Bigg|_0^{2\pi} && (14)\\
&= -\frac{1}{2}\left[\frac{1}{n+p}\cos(2\pi(n+p)) + \frac{1}{n-p}\cos(2\pi(n-p)) - \frac{1}{n+p} - \frac{1}{n-p}\right] && (15)\\
&= -\left[1-1\right] && (16)\\
&= 0
\end{aligned}
$$

where we have assumed $p \neq n$. If $p = n$, then the sine integration in (13) just goes to zero. We note that $\cos(2\pi n)$ will always be 1 in (15) since this is the beginning of it's period. A general identity which was derived above was

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]. \qquad (17)$$

□

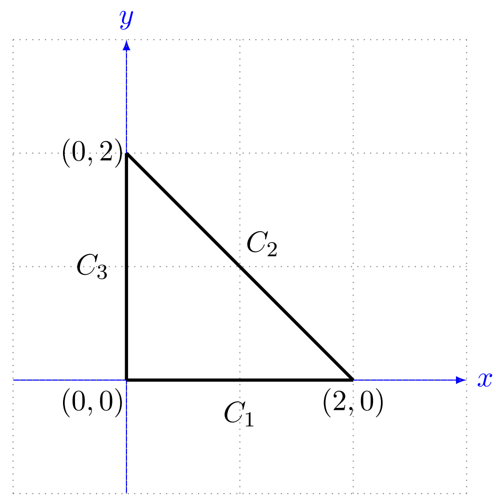
*Note: you will not be tested on this - I just wanted to provide proof of my claim.*

8. Let $\mathbf{V}(x,y) = \langle xy, 2y^2\rangle$ be a vector field, and let $A$ be the area bounded by the triangle $T = \{(x,y) \in \mathbb{R}^2 \,:\, 0 \le y \le 2-x,\ 0 \le x \le 2\}$. Show that Green's Theorem holds,

$$\oint_{\partial T} \mathbf{V} \cdot d\mathbf{r} = \iint_T (\nabla \times \mathbf{V}) \cdot d\mathbf{A}$$

where $\partial T$ denotes the boundary of $T$.

We begin with the line integral component of the theorem. The triangle has three sides, so we sum over 3 piecewise curves oriented anticlockwise.




We can parametrize the curves by the following functions, $y=0$, $y=2-x$, and $x=0$, hence obtaining

$$\mathbf{r}_1(t) = \langle t, 0\rangle \quad \text{for } \{0 \le t \le 2\} \tag{1}$$
$$\mathbf{r}_2(t) = \langle 2-t, t\rangle \quad \text{for } \{0 \le t \le 2\} \tag{2}$$
$$\mathbf{r}_3(t) = \langle 0, 2-t\rangle \quad \text{for } \{0 \le t \le 2\} \tag{3}$$

where we must have $C = C_1 \cup C_2 \cup C_3$ be oriented anticlockwise. This implies that we have

$$\mathbf{r}_1'(t) = \langle 1, 0\rangle \quad \text{for } \{0 \le t \le 2\} \tag{4}$$
$$\mathbf{r}_2'(t) = \langle -1, 1\rangle \quad \text{for } \{0 \le t \le 2\} \tag{5}$$
$$\mathbf{r}_3'(t) = \langle 0, -1\rangle \quad \text{for } \{0 \le t \le 2\} \tag{6}$$

From the definition of line integral, we compute the line integral over all three curves:

$$\int_{C_1} \mathbf{V} \cdot d\mathbf{r} = \int_0^2 \langle (t)(0), 2(0)^2\rangle \cdot \langle 1, 0\rangle\, dt \tag{7}$$
$$= 0 \tag{8}$$
$$\int_{C_2} \mathbf{V} \cdot d\mathbf{r} = \int_0^2 \langle 2t-t^2, 2t^2\rangle \cdot \langle -1, 1\rangle\, dt \tag{9}$$
$$= \int_0^2 \left[t^2 - 2t + 2t^2\right]\, dt \tag{10}$$

$$= (t^3 - t^2)\Big|_0^2 \tag{11}$$

$$= 4 \tag{12}$$

$$\int_{C_3} \mathbf{V} \cdot d\mathbf{r} = \int_0^2 \langle (0)(2-t), 2(2-t)^2 \rangle \cdot \langle 0, -1 \rangle \, dt \tag{13}$$

$$= -2 \int_0^2 (4 - 4t + t^2) \, dt \tag{14}$$

$$= -2 \left[ 4t - 2t^2 + \frac{1}{3} t^3 \right] \Big|_0^2 \tag{15}$$

$$= -\frac{16}{3} \tag{16}$$

and thus from (8), (12), and (16) we find that

$$\oint_C \mathbf{V} \cdot d\mathbf{r} = -\frac{4}{3} \tag{17}$$

The other side of Green's theorem contains the two-dimensional curl, from which becomes $\iint_T \left[ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right] dA$. We find

$$\iint_T \left[ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right] dA = \iint_T [0 - x] \; dA \tag{18}$$

$$= -\int_0^2 \int_0^{2-x} x \, dy dx \tag{19}$$

$$= -\int_0^2 (2x - x^2) \, dx \tag{20}$$

$$= -\left[ x^2 - \frac{1}{3} x^3 \right] \Big|_0^2 \tag{21}$$

$$= -\left[ 4 - \frac{8}{3} \right] \tag{22}$$

$$= -\frac{4}{3} \tag{23}$$

which matches (17) and thus Green's theorem holds.