MAT235 Tutorial 14: Solutions to Additional Problems

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Tutorial this week will focus on reviewing for Term Test 4 (March 12). There is no quiz. The first four problems will be taken up during tutorial time, while the additional six problems are for practice and whose solutions will be released prior to the test date.

1. Let $\mathbf{F}(x, y, z) = \langle -z, y, -x \rangle$ be a vector field.
   (a) Check that $\mathbf{F}(x, y, z)$ is conservative.
   (b) Find the scalar potential function $\varphi : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{F}(x, y, z) = \nabla\varphi(x, y, z)$.
   (c) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ using the definition of a line integral, where C is the curve parametrized by $\mathbf{r}(t) = \langle t, t^2, \cos t \rangle$ for $0 \le t \le \pi$.
   (d) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ using the Fundamental Theorem of Line Integrals.

2. Take the curl of each of the following vector fields $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ to determine whether or not the field is conservative. If it is conservative, find the potential function for the field.
   (a) $\mathbf{F}(x, y, z) = z\mathbf{j} + z\mathbf{k}$.
   (b) $\mathbf{F}(x, y, z) = \langle yz, xz, xy \rangle$.

3. Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = xy^2\mathbf{i} + \cos x\mathbf{j}$ and C is the boundary of the area bounded by the functions $y = |x|$ and $y = 1$, oriented anti-clockwise.

4. Consider the curve C defined by the implicit equation $x^{2/3} + y^{2/3} = a^{2/3}$ for some $a \in \mathbb{R}$. C can be parametrized by $x(t) = a\cos^3 t, y(t) = a\sin^3 t$ for $t \in [0, 2\pi]$. Use Green's theorem to show that the area bounded by C is $\frac{3}{8}a^2\pi$ given that $\int_0^{2\pi} \sin^2 x \cos^4 x \, dx = \frac{\pi}{8}$.

Additional Problems:

5. Consider the parallelogram $P = \{(x, y) \in \mathbb{R}^2 : 0 \le x - y \le 3, 1 \le x + y \le 3\}$ and its mass-density function $\delta : P \to \mathbb{R}$ given by $\delta(x, y) = (x + y)^2$. Use a change of variables to find the mass of P.

6. Evaluate $\iint_R (2x - \pi y) \, dA$ where R is the parallelogram with vertices $(2, 0), (5, 3), (6, 7), (3, 4)$. *Hint: derive a change of variables.*

7. The parametrization of a hypocycloid is given by

$$x(t) = (a - b)\cos t + b\cos\left(\frac{a - b}{b}\right)t$$

$$y(t) = (a - b)\sin t - b\sin\left(\frac{a - b}{b}\right)t$$

Let C_H be the 5-point hypocycloid curve given by parameters $a = 10$ and $b = 2$, and $0 \leq t \leq 2\pi$. Let $\mathbf{G} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field $\mathbf{G}(x, y) = \langle x, 2y \rangle$. Compute $\oint_{C_H} \mathbf{G} \cdot d\mathbf{r}$.

8. Let $\mathbf{V}(x, y) = \langle xy, 2y^2 \rangle$ be a vector field, and let A be the area bounded by the triangle $T = \{(x, y) \in \mathbb{R}^2 \,:\, 0 \leq y \leq 2 - x,\ 0 \leq x \leq 2\}$. Show that Green's Theorem holds,

$$\oint_{\partial T} \mathbf{V} \cdot d\mathbf{r} = \iint_T (\nabla \times \mathbf{V}) \cdot d\mathbf{A}$$

where ∂T denotes the boundary of T.