MAT235 Additional Problems in Vector Calculus: Solutions

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1. Let T be a torus centered at the origin of major radius $r_1 = 6$ in the xy-plane and minor radius $r_2 = 2$.

   (a) Parametrize T.

Solution: A torus can be constructed by wrapping a circle around a point (like a bent cylinder). We must therefore sum two different parametrizations: one of the circle of major radius r_1, and another of the circle of minor radius r_2.

Centered at the origin, the larger circle is simply parametrized by a curve of one variable,

$$\boldsymbol{\gamma}(\theta) = \langle r_1 \cos\theta, r_1 \sin\theta, 0\rangle \tag{1}$$

in $\mathbb{R}^3$ for $\theta \in [0, 2\pi]$. The challenge arises in parametrizing the smaller circle which wraps around the larger. If the torus lies in the xy plane, the smaller circle must sit upright, whose z-component is non-zero. The orientation of the smaller circle is thus dependent on the parametrization variable for the major circle. Breaking it down, we can first consider two circles of radius r_2 at an arbitrary point in the xy-plane $(a, b, 0)$ whose respective orientations are in the xz and yz planes. These are known parametrizations:

$$\mathbf{c}_1(\varphi) = \langle a + r_2 \cos\varphi,\ b,\ r_2 \sin\varphi\rangle \quad \text{for} \quad \varphi \in [0, 2\pi] \tag{2}$$

$$\mathbf{c}_2(\varphi) = \langle a,\ b + r_2 \cos\varphi,\ r_2 \sin\varphi\rangle \quad \text{for} \quad \varphi \in [0, 2\pi] \tag{3}$$

Here we observe that the only thing that changed what the shift of the $r_1 \cos\varphi$ term from the x-term in one orientation to the y-term in the second orientation. This is equivalent to multiplying $r_1 \cos\varphi$ by a parameter which fluctuates between 0 and 1:

$$\mathbf{c}(\varphi, \lambda) = \langle a + r_2 \cos\varphi \cos\lambda,\ b + r_2 \cos\varphi \sin\lambda, r_2 \sin\varphi\rangle \tag{4}$$

from which we notice that $\mathbf{c}_1(\varphi) = \mathbf{c}(\varphi, 0)$ and $\mathbf{c}_2(\varphi) = \mathbf{c}(\varphi, \pi/2)$. The parameter λ is now an orientation parameter, which allows one to decide which direction the circle is oriented in.

λ however, is the same parameter as θ in (1), since θ must specify the orientation of the minor circle. Furthermore, the location of $\mathbf{c}$ is (a, b), which is also given by (1), since (a,b) is equivalently the required endpoint of γ to loop the circle around the major axis. Thus we have

$$\mathbf{r}(\theta, \varphi) = \langle r_1 \cos\theta, r_1 \sin\theta, 0\rangle + \langle r_2 \cos\varphi \cos\theta,\ r_2 \cos\varphi \sin\theta, r_2 \sin\varphi\rangle \tag{5}$$

$$= \langle (r_1 + r_2 \cos\varphi) \cos\theta,\ (r_1 + r_2 \cos\varphi) \sin\theta,\ r_2 \sin\varphi\rangle \tag{6}$$

both for $\theta, \varphi \in [0, 2\pi]$, and this is the parametrization of the torus. For $r_1 = 6$ and $r_2 = 2$,

$$\mathbf{r}(\theta, \varphi) = \langle (6 + 2\cos\varphi) \cos\theta,\ (6 + 2\cos\varphi) \sin\theta,\ 2 \sin\varphi\rangle \quad \text{for} \quad \theta, \varphi \in [0, 2\pi]. \tag{7}$$

(b) Calculate the surface area of T.

Solution: We use the definition of surface area,

$$A = \iint_S \left\| \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} \right\| dA. \tag{8}$$

We use the parametrization in (6) then plug in $r_1 = 6$ and $r_2 = 2$. Begin first by calculating partial derivatives:

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial \theta} &= \langle -(r_1 + r_2 \cos\varphi)\sin\theta,\ (r_1 + r_2\cos\varphi)\cos\theta, 0\rangle & (9) \\ \frac{\partial \mathbf{r}}{\partial \varphi} &= \langle -r_2 \sin\varphi\cos\theta,\ -r_2\sin\varphi\sin\theta, r_2\cos\varphi\rangle & (10) \end{aligned}$$

Their cross product is then

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} &= \hat{\mathbf{x}}\,[(r_1 + r_2\cos\varphi)\cos\theta \cdot r_2\cos\varphi - (0)] + \hat{\mathbf{y}}\,[(0) + (r_1 + r_2\cos\varphi)\sin\theta \cdot r_2\cos\varphi] \\ &\quad + \hat{\mathbf{z}}\,[-(r_1 + r_2\cos\varphi)\sin\theta(-r_2\sin\varphi\sin\theta) - (r_1 + r_2\cos\varphi)\cos\theta(-r_2\sin\varphi\cos\theta)] & (11) \\ &= \hat{\mathbf{x}}\,r_2(r_1 + r_2\cos\varphi)\cos\theta\cos\varphi + \hat{\mathbf{y}}\,r_2(r_1 + r_2\cos\varphi)\sin\theta\cos\varphi \\ &\quad + \hat{\mathbf{z}}\,r_2(r_1 + r_2\cos\varphi)\sin\varphi(\sin^2\theta + \cos^2\theta) & (12) \\ &= \hat{\mathbf{x}}\,r_2(r_1 + r_2\cos\varphi)\cos\theta\cos\varphi + \hat{\mathbf{y}}\,r_2(r_1 + r_2\cos\varphi)\sin\theta\cos\varphi \\ &\quad + \hat{\mathbf{z}}\,r_2(r_1 + r_2\cos\varphi)\sin\varphi & (13) \\ &= r_2(r_1 + r_2\cos\varphi)\,[\hat{\mathbf{x}}\,\cos\theta\cos\varphi + \hat{\mathbf{y}}\,\sin\theta\cos\varphi + \hat{\mathbf{z}}\,\sin\varphi] & (14) \end{aligned}$$

Taking the vector magnitude of (14), we have

$$\begin{aligned} \left\| \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} \right\| &= \left[r_2^2(r_1 + r_2\cos\varphi)^2 \left(\cos^2\theta\cos^2\varphi + \sin^2\theta\cos^2\varphi + \sin^2\varphi\right)\right]^{1/2} & (15) \\ &= \left[r_2^2(r_1 + r_2\cos\varphi)^2 \left(\cos^2\varphi + \sin^2\varphi\right)\right]^{1/2} & (16) \\ &= \left[r_2^2(r_1 + r_2\cos\varphi)^2\,(1)\right]^{1/2} & (17) \\ &= r_2(r_1 + r_2\cos\varphi). & (18) \end{aligned}$$

The bounds of θ and φ have been specified in (a) when the parametrization was determined, so by definition the surface area of T is

$$\begin{aligned} A &= \int_0^{2\pi}\int_0^{2\pi} r_2(r_1 + r_2\cos\varphi)\,d\theta d\varphi & (19) \\ &= 2\pi \int_0^{2\pi} (r_1 r_2 + r_2^2\cos\varphi)\,d\varphi & (20) \\ &= (2\pi)^2 r_1 r_2 + (0) & (21) \\ &= 4\pi^2 r_1 r_2. & (22) \end{aligned}$$

Then, plugging in $r_1 = 6$ and $r_2 = 2$ for T, we have that the surface area of T is $48\pi^2$.

(c) Let $\mathbf{F}(x,y,z) = \langle 2x, 3y, -z\rangle$. Show that the divergence theorem holds for $\mathbf{F}$ fluxing through T.

Solution: Recall that the divergence theorem is given by $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_V (\nabla\cdot\mathbf{F})\,dV$. We will show that the left hand side is equal to the right hand side.

First, to calculate the flux, we begin by noting that the normal vector is given by (14) for an arbitrary torus of major and minor radii r_1 and r_2, respectively. We have that

$$\begin{aligned}\iint_T \mathbf{F}\cdot d\mathbf{S} &= \iint_T \mathbf{F}\cdot\mathbf{n}\,dS &(23)\\ &= \int_0^{2\pi}\int_0^{2\pi} \mathbf{F}(\mathbf{r}(\theta,\varphi))\cdot\left[\frac{\partial\mathbf{r}}{\partial\theta}\times\frac{\partial\mathbf{r}}{\partial\varphi}\right]d\theta d\varphi. &(24)\end{aligned}$$

To calculate the integrand, we plug in the parametrization from (6) and note that

$$\mathbf{F}(\mathbf{r}(\theta,\varphi)) = \langle 2(r_1+r_2\cos\varphi)\cos\theta,\ 3(r_1+r_2\cos\varphi)\sin\theta,\ -r_2\sin\varphi\rangle, \tag{25}$$

from which we take the dot product with (14),

$$\begin{aligned}\mathbf{F}(\mathbf{r}(\theta,\varphi))\cdot\left[\frac{\partial\mathbf{r}}{\partial\theta}\times\frac{\partial\mathbf{r}}{\partial\varphi}\right] &= r_2(r_1+r_2\cos\varphi)\Big[2(r_1+r_2\cos\varphi)\cos^2\theta\cos\varphi\\ &\qquad + 3(r_1+r_2\cos\varphi)\sin^2\theta\cos\varphi - r_2\sin^2\varphi\Big] &(26)\\ &= (r_2r_1+r_2^2\cos\varphi)\Big[2r_1\cos\varphi\cos^2\theta + 2r_2\cos^2\varphi\cos^2\theta\\ &\qquad + 3r_1\sin^2\theta\cos\varphi + 3r_2\cos^2\varphi\sin^2\theta - r_2\sin^2\varphi\Big] &(27)\\ &= 2r_1^2r_2\cos\varphi\cos^2\theta + 2r_1r_2^2\cos^2\varphi\cos^2\theta\\ &\quad + 3r_1^2r_2\sin^2\theta\cos\varphi + 3r_1r_2^2\cos^2\varphi\sin^2\theta - r_1r_2^2\sin^2\varphi\\ &\quad + 2r_1r_2^2\cos^2\varphi\cos^2\theta + 2r_2^3\cos^3\varphi\cos^2\theta\\ &\quad + 3r_1r_2^2\sin^2\theta\cos^2\varphi + 3r_2^3\cos^3\varphi\sin^2\theta - r_2^3\cos\varphi\sin^2\varphi. &(28)\end{aligned}$$

Now, this may seem complicated, but we have to remember that we are integrating over $[0,2\pi]$ in both θ and φ, so many of these terms will drop out. Recall that

$$\begin{aligned}\int_0^{2\pi}\sin x\,dx &= \int_0^{2\pi}\cos x\,dx = 0 &(29)\\ \int_0^{2\pi}\sin^2 x\,dx &= \int_0^{2\pi}\cos^2 x\,dx = \pi &(30)\\ \int_0^{2\pi}\sin^3 x\,dx &= \int_0^{2\pi}\cos^3 x\,dx = 0 &(31)\end{aligned}$$

where the second integral can be calculated using the identities $\sin^2 x = \frac{1}{2}(1-\cos(2x))$, $\cos^2 x = \frac{1}{2}(1+\cos(2x))$. Removing any trigonometric terms in θ or φ which are odd in sin or cos, (28) reduces to

$$\mathbf{F}(\mathbf{r}(\theta,\varphi))\cdot\left[\frac{\partial\mathbf{r}}{\partial\theta}\times\frac{\partial\mathbf{r}}{\partial\varphi}\right] = 2\cdot 2r_1r_2^2\cos^2\varphi\cos^2\theta + 2\cdot 3r_1r_2^2\sin^2\theta\cos^2\varphi - r_1r_2^2\sin^2\varphi. \tag{32}$$

Taking out the integral and using (30), we have

$$\iint_T \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{2\pi} (4r_1 r_2^2 \cos^2\varphi \cos^2\theta + 6r_1 r_2^2 \sin^2\theta \cos^2\varphi - r_1 r_2^2 \sin^2\varphi) \tag{33}$$

$$= 4r_1 r_2^2 \pi^2 + 6r_1 r_2^2 \pi^2 - (2\pi) r_1 r_2^2 \pi \tag{34}$$

$$= 10\pi^2 r_1 r_2^2 - 2\pi^2 r_1 r_2^2 \tag{35}$$

$$= 8\pi^2 r_1 r_2^2. \tag{36}$$

For $r_1 = 6$ and $r_2 = 2$, we have that the flux is $196\pi^2$. We can now take out the right hand side of the divergence theorem. The divergence of $\mathbf{F}$ is simply

$$\nabla \cdot \mathbf{F} = 2 + 3 - 1 = 4 \tag{37}$$

which is constant, hence the right hand side is just

$$4 \iiint_V dV = 4 \cdot \mathrm{vol}(T). \tag{38}$$

The volume of a torus is simple to calculate, as it is the area of a minor circle radius r_2 times the circumference of the major circle radius r_1, which is $(2\pi r_1)(\pi r_2^2) = 2\pi^2 r_1 r_2^2$. Therefore the right hand side is

$$\iiint (\nabla \cdot \mathbf{F})\, dV = 4 \cdot 2\pi^2 r_1 r_2^2 = 8\pi^2 r_1 r_2^2 \tag{39}$$

which is equivalent to the flux in (36). Therefore the divergence theorem holds and the flux is $196\pi^2$ for our torus of radii $r_1 = 6$ and $r_2 = 2$.

2. Consider a gravitational field specified by $\mathbf{G}(x, y, z) = \langle x^7 + xy^2z^2,\, x^2yz^2,\, x^2y^2z \rangle$. A drive flies in the field along a path C parametrized by $r(t) = \langle t\cos t,\, t\sin t,\, t(5\pi - t) \rangle$ for $t \in [0, 5\pi]$. Calculate the work $\displaystyle\int_C \mathbf{G} \cdot d\mathbf{r}$.

Solution: We compute the work from the definition of the line integral. First observe that the curl of $\mathbf{G}$ is $\mathbf{0}$:

$$\begin{aligned} \nabla \times \mathbf{G} &= \hat{\mathbf{x}}(\partial_y G_z - \partial_z G_y) + \hat{\mathbf{y}}(\partial_z G_x - \partial_x G_z) + \hat{\mathbf{z}}(\partial_x G_y - \partial_y G_x) && (1)\\ &= \hat{\mathbf{x}}(2x^2yz - 2x^2yz) + \hat{\mathbf{y}}(2xy^2z - 2xy^2z) + \hat{\mathbf{z}}(2xyz^2 - 2xyz^2) && (2)\\ &= \mathbf{0}. && (3) \end{aligned}$$

This implies that there exists a scalar field $g : \mathbb{R}^3 \to \mathbb{R}$ such that $\nabla g = \mathbf{G}$. Integrating each component of $\mathbf{G}$ respective to it's own variable, we have

$$\begin{aligned} g(x, y, z) &= \int (x^7 + xy^2z^2)\, dx && (4)\\ &= \frac{1}{8}x^8 + \frac{1}{2}x^2y^2z^2 + a(y, z) && (5)\\ g(x, y, z) &= \int x^2yz^2\, dy && (6)\\ &= \frac{1}{2}x^2y^2z^2 + b(x, z) && (7)\\ g(x, y, z) &= \int x^2y^2z\, dz && (8)\\ &= \frac{1}{2}x^2y^2z^2 + c(x, y) && (9) \end{aligned}$$

where a, b, c are scalar functions accounting for the integration constants which vanish upon partial differentiation. We observe that the overlapping term is $\frac{1}{2}x^2y^2z^2$, while $a(x, y) = 0$ and $b(x, y) = c(x, y) = \frac{1}{8}x^8$. Therefore the potential function for $\mathbf{G}$ is

$$g(x, y, z) = \frac{1}{2}x^2y^2z^2 + \frac{1}{8}x^8 \qquad (10)$$

From here, we can apply the fundamental theorem of line integrals,

$$\int_{\mathbf{r}(t=0)}^{\mathbf{r}(t=5\pi)} (\nabla g) \cdot d\mathbf{r} = g(\mathbf{r}(5\pi)) - g(\mathbf{r}(0)). \qquad (11)$$

Now $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$ and $\mathbf{r}(5\pi) = \langle 5\pi\cos(5\pi), 5\pi\sin(5\pi), 5\pi(5\pi - 5\pi) \rangle = \langle -5\pi, 0, 0 \rangle$. Thus

$$\begin{aligned} \int_C \mathbf{G} \cdot d\mathbf{r} &= \frac{1}{8}(-5\pi)^8 + \frac{1}{2}(5\pi)^2 \cdot (0) - (0) && (12)\\ &= \frac{(5\pi)^8}{8} && (13)\\ &\approx 4.63 \times 10^8. && (14) \end{aligned}$$

3. Let $\mathbf{F}(x, y, z) = \langle -y^2, x, z^2 \rangle$ be a vector field and let C be the curve defined by the intersection between the cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 9\}$ and the plane $\{(x, y, z) \in \mathbb{R}^3 : y + z = 5\}$. Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

Solution: Begin by noting that the curve C is a closed curve, since it is defined by the intersection between the cylinder of radius 3 along the z-axis and a plane:

$$C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 9\} \cap \{(x, y, z) \in \mathbb{R}^3 : y + z = 5\}. \tag{1}$$

We can choose the orientation of C so that it is a simple closed curve, from which we can apply Stoke's theorem to calculate the line integral. Recall that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS \tag{2}$$

where S is now the surface whose boundary is C. Start by finding the curl of $\mathbf{F}$:

$$\begin{aligned} \nabla \times \mathbf{F} &= \hat{\mathbf{x}}[\partial_y F_z - \partial_z F_y] + \hat{\mathbf{y}}[\partial_z F_x - \partial_x F_z] + \hat{\mathbf{z}}[\partial_x F_y - \partial_y F_x] && (3)\\ &= \hat{\mathbf{x}}[\partial_y(z^2) - \partial_z(x)] + \hat{\mathbf{y}}[\partial_z(-y^2) - \partial_x(z^2)] + \hat{\mathbf{z}}[\partial_x(x) - \partial_y(-y^2)] && (4)\\ &= \hat{\mathbf{z}}[1 + 2y]. && (5) \end{aligned}$$

We can now determine a parametrization of the surface of the intersection, which is just the plane $y + z = 5$ whose boundary is the circle of radius 3. We can parametrize by letting $x = s$, $y = t$ and having $z = 5 - y = 5 - t$. In cartesian coordinates, this is

$$\mathbf{r}(s, t) = \langle s, t, 5 - t \rangle \tag{6}$$

for $s, t \in \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$ but the symmetry of the problem suggests we use polar coordinates, so instead we can let $x = s = r\cos\theta$, $y = t = r\sin\theta$ so $z = 5 - r\sin\theta$ where $r \in [0, 3]$ and $\theta \in [0, 2\pi]$. We then have that

$$\mathbf{r}(r, \theta) = \langle r\cos\theta,\, r\sin\theta,\, 5 - r\sin\theta \rangle \quad \text{for} \quad r \in [0, 3],\ \theta \in [0, 2\pi]. \tag{7}$$

We next determine the normal vector for the parametrized surface:

$$\begin{aligned} \mathbf{n} = \frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} &= \langle \cos\theta,\, \sin\theta,\, -\sin\theta \rangle \times \langle -r\sin\theta,\, r\cos\theta,\, -r\cos\theta \rangle && (8)\\ &= \hat{\mathbf{x}}[-r\sin\theta\cos\theta + r\sin\theta\cos\theta] + \hat{\mathbf{y}}[r\sin^2\theta + r\cos^2\theta] + \hat{\mathbf{z}}[r\cos^2\theta + r\sin^2\theta] && (9)\\ &= r\langle 0,\, 1,\, 1 \rangle. && (10) \end{aligned}$$

We can now proceed by computing the flux of the curl through the surface:

$$\begin{aligned} \iint_S (\nabla \times \mathbf{F})(\mathbf{r}(r, \theta)) \cdot \mathbf{n}\, dS &= \int_0^3 \int_0^{2\pi} \langle 0,\, 0,\, 1 + 2r\sin\theta \rangle \cdot \langle 0,\, 1,\, 1 \rangle\, r d\theta dr && (11)\\ &= \int_0^3 \int_0^{2\pi} (1 + 2r\sin\theta)\, r d\theta dr && (12)\\ &= \int_0^3 r dr \int_0^{2\pi} d\theta + 2\int_0^3 r^2\, dr \int_0^{2\pi} \sin\theta\, d\theta && (13)\\ &= \frac{1}{2}r^2\Big|_0^3 \cdot (2\pi) + (0) && (14)\\ &= 9\pi. && (15) \end{aligned}$$

By Stoke's theorem, we have that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 9\pi$.

4. Consider two arbitrary, second order continuously differentiable vector fields $\mathbf{A}$ and $\mathbf{B}$ in $\mathbb{R}^3$. Prove that

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B}).$$

Proof: Consider two arbitrary, C^2 vector fields $\mathbf{A} = \langle A_x, A_y, A_z \rangle$, and $\mathbf{B} = \langle B_x, B_y, B_z \rangle$. We have that

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{x}}(A_y B_z - A_z B_y) + \hat{\mathbf{y}}(A_z B_x - A_x B_z) + \hat{\mathbf{z}}(A_x B_y - A_y B_x) \tag{1}$$

so the divergence is, noting that $\mathbf{A}$ and $\mathbf{B}$ are second order continuously differentiable fields and we can take derivatives of any components without regarding any flaw of discontinuity,

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \partial_x(A_y B_z - A_z B_y) + \partial_y(A_z B_x - A_x B_z) + \partial_z(A_x B_y - A_y B_x). \tag{2}$$

Expanding (2) out using the product rule,

$$\begin{aligned}
\nabla \cdot (\mathbf{A} \times \mathbf{B}) &= B_z[\partial_x A_y] + A_y[\partial_x B_z] - B_y[\partial_x A_z] - A_z[\partial_x B_y] \\
&\quad + B_x[\partial_y A_z] + A_z[\partial_y B_x] - B_z[\partial_y A_x] - A_x[\partial_y B_z] \\
&\quad + B_y[\partial_z A_x] + A_x[\partial_z B_y] - B_x[\partial_z A_y] - A_y[\partial_z B_x] && (3) \\
&= B_x(\partial_y A_z - \partial_z A_y) + B_y(\partial_z A_x - \partial_x A_z) + B_z(\partial_x A_y - \partial_y A_x) \\
&\quad + A_x(\partial_z B_y - \partial_y B_z) + A_y(\partial_x A_y - \partial_y A_x) + A_z(\partial_y B_x - \partial_x B_y) && (4) \\
&= B_x(\partial_y A_z - \partial_z A_y) + B_y(\partial_z A_x - \partial_x A_z) + B_z(\partial_x A_y - \partial_y A_x) \\
&\quad - A_x(\partial_y B_z - \partial_z B_y) - A_y(\partial_y A_x - \partial_x A_y) - A_z(\partial_x B_y - \partial_y B_x) && (5)
\end{aligned}$$

We notice that the first line in (5) contains the components of the curl of $\mathbf{A}$ dotted with the components of $\mathbf{B}$, while the second line contains the (negative) components of the curl of $\mathbf{B}$ dotted with the components of $\mathbf{A}$, hence we have

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \tag{6}$$

but since the dot product is commutative,

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B}). \tag{7}$$

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5. Find the positively oriented, simple closed curve L such that the line integral

$$\oint_L (y^3 - y)\, dx - 2x^3\, dy$$

is a maximum.

Solution: We consider the line integral $\oint_L (y^3 - y)\, dx - 2x^3\, dy$, which can be represented by the dot product between the field $\mathbf{F}(x, y) = \langle y^3 - y,\, 2x^3\rangle$ and the displacement vector $d\mathbf{r} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy$. We therefore have

$$\oint_L \langle y^3 - y,\, -2x^3\rangle \cdot d\mathbf{r}. \tag{1}$$

It is straightforward to check that $\nabla \times \mathbf{F} \neq \mathbf{0}$, so we cannot write $\mathbf{F}$ as a gradient field and apply the fundamental theorem of line integrals. Instead, since we require L to be a positively oriented, simple closed curve, we can apply Stoke's theorem

$$\oint_{\partial S} \mathbf{G} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{G}) \cdot d\mathbf{S}. \tag{2}$$

The curl of $\mathbf{F}$ is oriented along the $\mathbf{z}$ axis (and so is our area element), hence we have

$$(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{z}} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \tag{3}$$

$$= -6x^2 - 3y^2 + 1 \tag{4}$$

Hence

$$\oint_L \langle y^3 - y,\, -2x^3\rangle \cdot d\mathbf{r} = \iint_A (-6x^2 - 3y^2 + 1)\, dA \tag{5}$$

so we must find the set A which maximizes the area integral in (5). Now, A is just a set in $\mathbb{R}^2$, and $(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{z}}$ is real-valued function (which is an inverted paraboloid), we can just find the restriction of $\mathbb{R}^2$ where the integrand is strictly positive. Finding the intersection of $-6x^2 - 3y^2 + 1$ with the $z = 0$ plane then specifies the boundary between positive and negative outputs, from which we can determine A.

$-6x^2 - 3y^2 + 1 = 0$ implies that the values (x, y) outside of this region correspond to negative values of the integrand, and (x, y) points inside this region correspond to positive values of the integrand. Hence A becomes

$$A = \{(x, y) \in \mathbb{R}^2 \,:\, 6x^2 + 3y^2 \leq 1\}. \tag{6}$$

This is therefore the set which maximizes (5), whose boundary is the positively oriented simple closed curve which maximizes (1). This boundary is parametrized by

$$\mathbf{r}(t) = \left\langle \frac{1}{\sqrt{6}} \cos t,\, \frac{1}{\sqrt{3}} \sin t \right\rangle \quad \text{for} \quad t \in [0, 2\pi] \tag{7}$$

hence

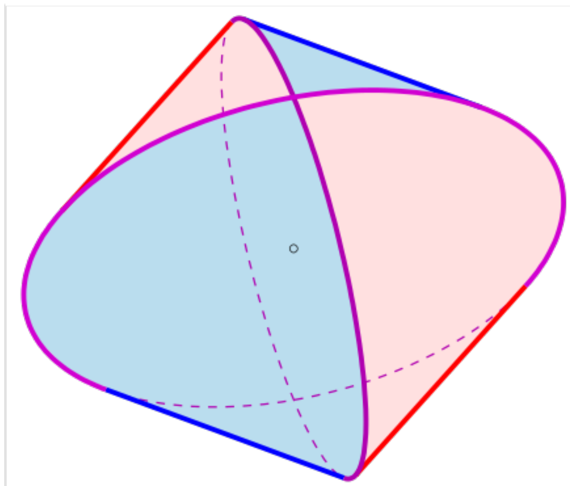
$$L = \{(x, y) \in \mathbb{R}^2 \,:\, 6x^2 + 3y^2 = 1\}. \tag{8}$$

6. Calculate the volume of the intersection of two cylinders of radius r. From the volume, determine the surface area.

Solution: Losing no generality, let us choose two cylinder of radius r lying along the x and y axes, whose point of intersection is at the origin (0,0,0). That is, the set

$$Q = \{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + z^2 \leq r^2\} \cap \{(x, y, z) \in \mathbb{R}^3 \,:\, y^2 + z^2 \leq r^2\}. \tag{1}$$

To calculate the volume, we must calculate the bounds. Notice that Q represents the solid



which appears like a spherical-cuboid intersection. Although one may assume, by symmetry, for it to be easiest to use cylinderical coordinates, but by the cartestian symmetry when one looks down onto the shape in the xy plane, it may suggest to be best to use cartestian coordinates. This same argument implies that the orientation of the object along x or y is completely arbitrary, hence we can integrate small 'cubical' elements.

For instance, along the first cylinder, the range of distance from the axis of symmetry to the surface at any z value in Q is

$$\sqrt{x^2 + z^2} \in [0, r] \implies x \in [0, \sqrt{r^2 - z^2}\,]. \tag{1}$$

Though this accounts for the range in radius to this point, it only accounts for half of object since there are positive and negative values of x. Multiplying by two accounts for the other half, which is equivalently the diameter between two points on the surface opposite of each other:

$$2x \in [0, \sqrt{r^2 - z^2}\,]. \tag{2}$$

A simple change of variables gives the exact same diameter for the other cylinder at any z value in Q: $2y \in [0, \sqrt{r^2 - z^2}]$.

Hence, to find the whole volume, we multiply the ranges of the x components of the cuboid with those of the y cuboid, then integrate from $-r \leq z \leq r$. The area of a rectangle is $\ell \times w$, so if our length is $2x$ and width is $2y$, we have the area of a small cuboid at any z value with radius r, which is $(2x)(2y)$. We can then substitute in our bounds for x and y in terms of z, with z ranging from $[-r, r]$, and then we can integrate:

$$
\begin{aligned}
V &= \int_{-r}^{r} \int_{-x}^{x} \int_{-y}^{y} dx' dy' dz && (3)\\
&= \int_{-r}^{r} (2x)(2y)\, dz && (4)\\
&= 4 \int_{-r}^{r} (\sqrt{r^2 - z^2})(\sqrt{r^2 - z^2})\, dz && (5)\\
&= 4 \int_{-r}^{r} (r^2 - z^2)\, dz && (6)\\
&= 4 \left[2r^3 - \frac{2}{3} r^3 \right] && (7)\\
&= \frac{16}{3} r^3 && (8)
\end{aligned}
$$

which is the volume.

To determine the surface area, we can recall that the change in volume of an object with respect to its radius is the surface area of an object, since individual surface area 'shells' build up the volume. Hence

$$
\begin{aligned}
SA &= \frac{dV}{dr} && (9)\\
&= 16r^2. && (10)
\end{aligned}
$$

This is called a Steinmetz solid, and there are many other variations for intersections of 3, 4, or 5 cylinders at the same point.

7. Find parametric equations for each of the following objects and, where applicable, calculate their unit tangent vectors and unit normal vectors:

   (i) Sphere of radius 6 centered at $(2, 5, -\pi)$.

Solution: We use a parametrization equivalent to a change of variables for a sphere. We then shift the sphere to the point $(2, 5, -\pi)$. A sphere of radius R has a parametrization / change of variables transformation

$$\boldsymbol{\sigma}(\theta, \varphi) = \langle R \sin\varphi \cos\theta,\ R \sin\varphi \sin\theta,\ R \cos\varphi \rangle \quad \text{for} \quad \theta \in [0, 2\pi],\ \varphi \in [0, \pi]. \tag{1}$$

We take $R = 6$ and shift the sphere to the point $(2, 5, -\pi)$, so we have

$$\boldsymbol{\sigma}(\theta, \varphi) = \langle 6 \sin\varphi \cos\theta + 2,\ 6 \sin\varphi \sin\theta + 5,\ 6 \cos\varphi - \pi \rangle \quad \text{for} \quad \theta \in [0, 2\pi],\ \varphi \in [0, \pi] \tag{2}$$

for the parametrization.

Next we determine the normal vector to the surface. The tangent vectors are

$$\frac{\partial \boldsymbol{\sigma}}{\partial \theta} = \langle -6 \sin\varphi \sin\theta,\ 6 \sin\varphi \cos\theta, 0 \rangle \tag{3}$$

$$\frac{\partial \boldsymbol{\sigma}}{\partial \varphi} = \langle 6 \cos\varphi \cos\theta,\ 6 \cos\varphi \sin\theta,\ -6 \sin\varphi \rangle \tag{4}$$

from which the normal vector is

$$\begin{aligned} \frac{\partial \boldsymbol{\sigma}}{\partial \theta} \times \frac{\partial \boldsymbol{\sigma}}{\partial \varphi} &= \hat{\mathbf{x}}[-36 \sin^2\varphi \cos\theta] - \hat{\mathbf{y}}[36 \sin^2\varphi \sin\theta] \\ &\quad + \hat{\mathbf{z}}[-36 \sin\varphi \cos\varphi \sin^2\theta - 36 \sin\varphi \cos\varphi \cos^2\theta] \qquad (5) \\ &= 36 \sin\varphi \left[-\hat{\mathbf{x}}[\sin\varphi \cos\theta] - \hat{\mathbf{y}}[\sin\varphi \sin\theta] - \hat{\mathbf{z}}[\cos\varphi]\right] \qquad (6) \end{aligned}$$

Note that the negative sign implies the surface is positively oriented. There are two artificial singularities at $\varphi = 0, \pi$ and this arises due to the choice of parametrization. This is ok and can be fixed by normalizing the vector:

$$\hat{\mathbf{n}}(\theta, \varphi) = -\langle \sin\varphi \cos\theta,\ \sin\varphi \sin\theta,\ \cos\varphi \rangle \quad \text{for} \quad \theta \in [0, 2\pi],\ \ \varphi \in [0, \pi]. \tag{7}$$

(ii) Helix of radius 2 along the y-axis interval $[-3, 1]$ with 3 rotations every horizontal unit.

Solution: Recall that a helix of radius R along the z-axis, for instance, with one rotation every vertical unit can be given by

$$\boldsymbol{\gamma}(t) = \left\langle R\cos t,\, R\sin t,\, \frac{1}{2\pi}t \right\rangle \quad \text{for} \quad t \in \mathbb{R}. \tag{8}$$

The $\frac{1}{2\pi}$ is the rate of increase along the z axis because it requires 2π radians to cover one full rotation for a circle. We, however, must orient the helix so that it is along the y-axis. We can then determine the factor m governing rotations per unit along the y-axis and the bounds on t. We can start by rearranging (3) so that we travel along the y-axis, not the z:

$$\boldsymbol{\gamma}(t) = \left\langle R\cos t,\, mt,\, R\sin t \right\rangle. \tag{9}$$

For one rotation every unit, m must be $\frac{1}{2\pi}$ since it takes 2π rotations to cover one circle loop of the helix. If we think of m as a slope, the rise over run is $\frac{1}{1} \cdot \frac{1}{2\pi}$. For half a rotation every unit, we take the slope to be $\frac{2}{1}$ or $m = \frac{1}{\pi}$, so it takes two units to complete one rotation. Thus $m = \frac{\#\text{ units}}{\#\text{ rotations}} \cdot \frac{1}{2\pi}$, and for 2 rotations per 1 unit we have that $m = \frac{1}{2} \cdot \frac{1}{2\pi}$. Applying this concept to our helix, 3 rotations every horizontal unit along y implies that $m = \frac{1}{3}\frac{1}{2\pi} = \frac{1}{6\pi}$. For $R = 2$, our parametrization of our helix is then

$$\boldsymbol{\gamma}(t) = \left\langle 2\cos t,\, \frac{1}{6\pi}t,\, 2\sin t \right\rangle. \tag{10}$$

The bounds of t are still unknown $[t_1, t_2]$. We require $\boldsymbol{\gamma}$ to be along the y-interval $[-3, 1]$. So at the endpoints we match the y-components, $\frac{1}{6\pi}t_1 = -3$ and $\frac{1}{6\pi}t_2 = 1$. Rearranging for t implies that $t_1 = -18\pi$ and $t_2 = 6\pi$. Therefore

$$\boldsymbol{\gamma}(t) = \left\langle 2\cos t,\, \frac{1}{6\pi}t,\, 2\sin t \right\rangle \quad \text{for} \quad t \in [-18\pi, 6\pi]. \tag{11}$$

The tangent vector is found by just taking the derivative,

$$\boldsymbol{\gamma}'(t) = \left\langle -2\sin t,\, \frac{1}{6\pi},\, 2\cos t \right\rangle. \tag{12}$$

The magnitude of this vector is just $\sqrt{4\sin^2 t + 4\cos^2 t + \frac{1}{(6\pi)^2}}$ so we normalize it to the unit tangent vector,

$$\hat{\mathbf{T}}(t) = \frac{1}{\sqrt{1 + \frac{1}{(12\pi)^2}}} \left\langle -\sin t,\, \frac{1}{12\pi},\, \cos t \right\rangle \quad \text{for} \quad t \in [-18\pi, 6\pi] \tag{13}$$

(iii) Ellipse of minimum radius $\frac{1}{2}$ and maximum radius $\sqrt{2}$ centered at $(-2, 5)$.

Solution: Begin with the parametrization of a circle of radius 1 in $\mathbb{R}^2$, then we can modify it:

$$\mathbf{r}(t) = \langle \cos t,\, \sin t\rangle \quad \text{for} \quad t \in [0, 2\pi]. \tag{14}$$

Notice that $\mathbf{r}(0) = \langle 1, 0\rangle$ and $\mathbf{r}(\pi/2) = \langle 0,\, 1\rangle$. If we were to scale, say, the x-component by 2, I get an ellipse back:

$$\begin{aligned} \mathbf{r}(t) &= \langle 2\cos t, \sin t\rangle \quad \text{for} \quad t \in [0, 2\pi] && (15)\\ \Longrightarrow\ \mathbf{r}(0) &= \langle 2, 0\rangle && (16)\\ \Longrightarrow\ \mathbf{r}(\pi/2) &= \langle 0, 1\rangle. && (17) \end{aligned}$$

We can then just pick two scaling factors to scale our circle by, which will produce an ellipse of two different radii. Scaling the x axis by $\frac{1}{2}$ and the y by $\sqrt{2}$ (you can do this the other way around), we find that the parametrization of an ellipse at the origin is

$$\mathbf{r}(t) = \left\langle \frac{1}{2}\cos t, \sqrt{2}\sin t \right\rangle. \tag{18}$$

We can then just shift it to the point (-2, 5):

$$\mathbf{r}(t) = \left\langle \frac{1}{2}\cos t - 2, \sqrt{2}\sin t + 5 \right\rangle \quad \text{for} \quad t \in [0, 2\pi]. \tag{19}$$

An alternative solution is

$$\mathbf{r}(t) = \left\langle \sqrt{2}\cos t - 2, \frac{1}{2}\sin t + 5 \right\rangle \quad \text{for} \quad t \in [0, 2\pi]. \tag{20}$$

The tangent vector is just given by taking the derivative of the parametrization,

$$\mathbf{r}'(t) = \left\langle -\frac{1}{2}\sin t,\ \sqrt{2}\cos t \right\rangle \tag{21}$$

from which the magnitude is $\sqrt{\frac{1}{4}\sin^2 t + 2\cos^2 t} = \frac{1}{2}\sqrt{1 + 7\cos^2 t}$, hence the unit tangent vector is

$$\hat{\mathbf{T}}(t) = \frac{1}{\sqrt{1 + 7\cos^2 t}} \left\langle -\sin t,\ 2\sqrt{2}\cos t \right\rangle \quad \text{for} \quad t \in [0, 2\pi]. \tag{22}$$

(iv) The surface specified by $z = f(x,y) = x^3 - \log(y)$ on the domain $\{(x,y) \in \mathbb{R}^2 \,:\, y > 0\}$.

Solution: Letting $x = s$ and $y = t$, we write the parametrization as $\boldsymbol{\lambda}(s,t) = \langle s, t, f(s,t)\rangle$ for scalar functions. The bounds are already given by the set, and thus we have

$$\boldsymbol{\lambda}(s,t) = \langle s, t, s^3 - \log t\rangle \quad \text{for} \quad s \in \mathbb{R},\;\; t \in (0,\infty). \tag{23}$$

We lastly calculate the surface tangent vectors by taking derivatives,

$$\frac{\partial \boldsymbol{\lambda}}{\partial s} = \langle 1,\, 0,\, 3s^2\rangle \tag{24}$$

$$\frac{\partial \boldsymbol{\lambda}}{\partial t} = \left\langle 0,\, 1\, -\frac{1}{t}\right\rangle \tag{25}$$

from which the normal vector can be computed by cross product

$$\frac{\partial \boldsymbol{\lambda}}{\partial s} \times \frac{\partial \boldsymbol{\lambda}}{\partial t} = \hat{\mathbf{x}}[-3s^2] + \hat{\mathbf{y}}\left[\frac{1}{t}\right] + \hat{\mathbf{z}}[1] \tag{26}$$

$$= \left\langle -3s^2,\, \frac{1}{t},\, 1\right\rangle \tag{27}$$

whose magnitude is $\sqrt{9s^4 + \frac{1}{t^2} + 1}$, therefore the normalized normal vector is

$$\hat{\mathbf{T}}(s,t) = \left[9s^4 + \frac{1}{t^2} + 1\right]^{-1/2} \left\langle -3s^2,\, \frac{1}{t},\, 1\right\rangle \quad \text{for} \quad s \in \mathbb{R},\;\; t \in (0,\infty). \tag{28}$$