MAT235 Additional Problems in Vector Calculus

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1. Let T be a torus centered at the origin of major radius $r_1 = 6$ in the xy-plane and minor radius $r_2 = 2$.
   (a) Parametrize T.
   (b) Calculate the surface area of T.
   (c) Let $\mathbf{F}(x, y, z) = \langle 2x, 3y, -z\rangle$. Show that the divergence theorem holds for $\mathbf{F}$ fluxing through T.
2. Consider a gravitational field specified by $\mathbf{G}(x, y, z) = \langle x^7 + xy^2z^2,\, x^2yz^2,\, x^2y^2z\rangle$. A drive flies in the field along a path C parametrized by $r(t) = \langle t\cos t,\, t\sin t,\, t(5\pi - t)\rangle$ for $t \in [0, 5\pi]$. Calculate the work $\displaystyle\int_C \mathbf{G} \cdot d\mathbf{r}$.
3. Let $\mathbf{F}(x, y, z) = \langle -y^2, x, z^2\rangle$ be a vector field and let C be the curve defined by the intersection between the cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 9\}$ and the plane $\{(x, y, z) \in \mathbb{R}^3 : y + z = 5\}$. Compute $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r}$.
4. Consider two arbitrary, second order continuously differentiable vector fields $\mathbf{A}$ and $\mathbf{B}$ in $\mathbb{R}^3$. Prove that
$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B}).$$
5. Find the positively oriented, simple closed curve L such that the line integral
$$\oint_L (y^3 - y)\, dx - 2x^3\, dy$$
is a maximum.
6. Calculate the volume of the intersection of two cylinders of radius r. From the volume, determine the surface area.
7. Find parametric equations for each of the following objects and, where applicable, calculate their unit tangent vectors and unit normal vectors:
   - Sphere of radius 6 centered at $(2, 5, -\pi)$.
   - Helix of radius 2 along the y-axis interval $[-3, 1]$ with 3 rotations every horizontal unit.
   - Ellipse of minimum radius $\frac{1}{2}$ and maximum radius $\sqrt{2}$ centered at $(-2, 5)$.
   - The surface specified by $z = f(x, y) = x^3 - \log(y)$ on the domain $\{(x, y) \in \mathbb{R}^2 : y > 0\}$.

Note: For additional practice on volume integrals, Lagrange multipliers, multivariable Taylor expansions, or partial derivatives, I suggest visiting https://tutorial.math.lamar.edu/Problems/CalcIII/CalcIII.aspx*.*