MAT235 Tutorial 1

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This week: algebraic surfaces, completing the square, cross-sections. Refer 12.2, 12.3 Hughes-Hallett.

Review: Just as you have encountered common curves in single-variable calculus (lines, circles, ellipses etc.) there are three-dimensional extensions consisting much more complicated surfaces (hyper surfaces for $\mathbb{R}^n$ dimensions). We aim to express these algebraically:

- Paraboloid:
$$z = a(x^2 + y^2) \tag{1}$$
for a paraboloid opening in z-direction with some scaling factor $a \in \mathbb{R}$. Notice that at $y = 0$, the surface reduces to a parabola, like what you have seen in previous functions classes.

- Sphere of radius r centered at a point (x_0, y_0, z_0):
$$r^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2. \tag{2}$$

- Ellipsoid of major, semi-major, and minor axes A, B, C at a point (x_0, y_0, z_0):
$$1 = \left(\frac{x - x_0}{A}\right)^2 + \left(\frac{y - y_0}{B}\right)^2 + \left(\frac{z - z_0}{C}\right)^2. \tag{3}$$

- Plane passing through a point (x_0, y_0, z_0):
$$z = a(x - x_0) + b(y - y_0) + z_0, \tag{4}$$
where a and b represent the independent slopes along the x and y axes, respectively. Later in this course we will investigate other forms to express planes, namely using normal vectors, which can be determined from the standard form by taking a and b as vector components and computing a cross-product (if this doesn't sound familiar, don't worry - we'll visit this again in a few months).

Above are some general expressions in what is called 'standard form.' We will investigate other representations of these surfaces in the future, especially when we get to vector calculus, and begin looking at line and surface parametrizations. Sometimes, the standard forms of the surfaces are not given, and we must either expand or complete the square in each variable to obtain a recognizable expression. Since this can be done, given that each variable is indepent of the other (no 'xy'-like product terms), we can begin with reviewing the single-variable process:

1. Given the quadratic equation in the form $ax^2 + bx + c$, for general real numbers a, b, c, we aim to get this algebra into a 'standard form' quadratic expression. The first step would be to divide out the a to begin manipulating the other terms:
$$ax^2 + bx + c = a\left(x^2 + Bx + C\right) \tag{5}$$
where $B = \dfrac{b}{a}$ and $C = \dfrac{c}{a}$ are general re-scaling terms (obviously assuming $a \neq 0$).

2. We can now focus on the quadratic inside the paratheses. Recall that, in a quadratic expansion, $(\alpha+\beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$. Treating the '$B$' coefficient as the 2β term, and x as the α variable from the expansion, we can 'add zero' by including a $+\beta^2 - \beta^2 = \frac{1}{4}B^2 - \frac{1}{4}B^2$ (for the squared term).

$$x^2 + Bx + C = x^2 + Bx + \underbrace{\left(\frac{1}{2}B\right)^2 - \left(\frac{1}{2}B\right)^2}_{=0} + C \tag{6}$$

$$= \left[x^2 + Bx + \left(\frac{1}{2}B\right)^2\right] - \left(\frac{1}{2}B\right)^2 + C \tag{7}$$

3. Lastly, we can factor the terms inside the [] paratheses into the $(\alpha+\beta)^2$ standard form,

$$\left[x^2 + Bx + \left(\frac{1}{2}B\right)^2\right] - \left(\frac{1}{2}B\right)^2 + C = \left[x + \left(\frac{1}{2}B\right)\right]^2 - \left(\frac{1}{2}B\right)^2 + C \tag{8}$$

You can expand it out and verify it will give you back the initial expression $x^2 + Bx + C$. We can then re-introduce the a, b, c variable notation if necessary:

$$ax^2 + bx + c = a\left[x + \left(\frac{1}{2}\frac{b}{a}\right)\right]^2 - \frac{1}{4}\frac{b^2}{a} + c. \tag{9}$$

Note that the sign of b is irrelevant in the constant, since it is being squared and b is not a complex value. We can now implement this process on a multi-variable example.
Consider $x^2 + y^2 + z^2 - 6x + 4y + 10z = 0$. Re-arrange this expression into a standard form by completing the square in each variable:

$$x^2 + y^2 + z^2 - 6x + 4y + 10z = (x^2 - 6x) + (y^2 + 4y) + (z^2 + 10z) \tag{10}$$

$$= (x^2 - 6x + 9 - 9) + (y^2 + 4y + 4 - 4) + (z^2 + 10z + 25 - 25) \tag{11}$$

$$= (x^2 - 6x + 9) + (y^2 + 4y + 4) + (z^2 + 10z + 25) - 9 - 4 - 25 \tag{12}$$

$$= (x-3)^2 + (y+2)^2 + (z+5)^2 - 38 \tag{13}$$

$$\implies 38 = (x-3)^2 + (y+2)^2 + (z+5)^2. \tag{14}$$

This expression corresponds to a sphere of radius $\sqrt{38}$ centered at the point $(3, -2, -5)$. Note how I added and subtracted $\left(\frac{1}{2}b\right)^2$ in each variable, then took the b value as the translation along each respective axes.

Problems (solutions posted on main MAT235 Quercus; we'll also go over them in class):

1. Consider the sphere S defined by $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 - 6x + 4y + 10z = 0\}$.
   (a) Find the cross-sections of S when $x = 0$ and when $y = 1$. Describe them algebraically and in words.
   (b) Do the shapes of the cross-sections $x = c$ and $y = c$ (for a constant c) change as c is varied? If so, how?
2. Suppose there is a surface C in $\mathbb{R}^3$ whose non-empty level curves (level curves which are not outside of C) are always circles or single points. Find two expressions, representing different surfaces, which could possible represent C.