MAT235 Tutorial 10: Solutions to Additional Problems

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5. Show that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. *This is called the Gaussian integral, I. Hint: extend the integral into two-dimensions, then transform into polar coordinates. We can write*

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy.$$

Solution: Let $I = \int_{\infty}^{\infty} e^{-x^2}\,dx$. This is a single-variable integral, but we can extend the integral into two dimensions by squaring it:

$$I^2 = \int_{\infty}^{\infty} e^{-x^2}\,dx \int_{\infty}^{\infty} e^{-y^2}\,dy \tag{1}$$

Since the bounds of integration are variable-independent and are over all of two-dimensional orthogonal space ($\mathbb{R}^2$), we can write

$$I^2 = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dxdy \tag{2}$$

where the bounds of integration extend to infinity along each axis. Changing the integral into polar coordinates allows us to complete the integral by letting $r^2 = x^2 + y^2$, so $dxdy \to rd\theta dr$ (the bounds of integration are still over all space):

$$I^2 = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dxdy \tag{3}$$

$$= \int_0^{2\pi} \int_0^{\infty} e^{-r^2}\,rdrd\theta \tag{4}$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} re^{-r^2}\,dr \tag{5}$$

$$= (2\pi)\left(-\frac{1}{2}\lim_{r\to\infty}[e^{-r^2} - e^{(0)}]\right) \tag{6}$$

$$= \pi \tag{7}$$

Therefore $I^2 = \pi$, so $I = \pm\sqrt{\pi}$. However, since the Gaussian function is strictly positive $e^{-x^2} > 0$, the area of the integral must also be positive for any value of x, so we choose the positive value of I. Hence $I = \sqrt{\pi}$.

6. The plane $x - y + z = 2$ intersects the cylinder $x^2 + y^2 = 4$ in an ellipse. Find the points on the ellipse closest to and furthest from the origin. *Hint: to optimize distance* $\sqrt{x^2 + y^2 + z^2}$ *in* $\mathbb{R}^3$*, we can optimize* $f(x, y, z) = x^2 + y^2 + z^2$ *as it's derivative is simpler.*

Solution: To optimize distance in $\mathbb{R}^3$, we optimize $\sqrt{x^2 + y^2 + z^2}$ as given by the Pythagorean theorem. Suppose we have two distances a, b which are non-negative with $a < b$. Multiplying both sides by a and b independently give that $a^2 < ab$ and $ab < b^2$. Thus $a^2 < ab < b^2$, so $a^2 < b^2$. Therefore, to simplify the derivatives of the problem, we can optimize the square of the distance, the function $f(x, y, z) = x^2 + y^2 + z^2$ whose level surfaces are spheres in $\mathbb{R}^3$. This is a two constraint problem, since the gradient of the intersection (the ellipse) is <u>not</u> equal to the gradient of the plane and the cylinder together. Therefore we use

$$\nabla f = \lambda \nabla g + \mu \nabla h \tag{1}$$
$$g(x, y, z) = c \tag{2}$$
$$h(x, y, z) = k \tag{3}$$

to determine a system of equations whose constrains imply an optimization with respect to the intersection of both surfaces. Let $g(x, y, z) = x - y + z$ be the plane and $h(x, y, z) = x^2 + y^2$ be the cylinder along the z-axis. The gradients are

$$\nabla f(x, y, z) = \langle 2x, 2y, 2z \rangle \tag{4}$$
$$\nabla g(x, y, z) = \langle 1, -1, 1 \rangle \tag{5}$$
$$\nabla h(x, y, z) = \langle 2x, 2y, 0 \rangle. \tag{6}$$

Matching the x, y, z components respectively give a 5-equation system from (1), (2), (3) above:

$$2x = \lambda + 2\mu x \tag{7}$$
$$2y = -\lambda + 2\mu y \tag{8}$$
$$2z = \lambda \tag{9}$$
$$x - y + z = 2 \tag{10}$$
$$x^2 + y^2 = 4. \tag{11}$$

Let's begin by examining the implications of equations (7) and (8). Adding these equations imply that

$$x + y = \mu(x + y) \tag{12}$$

which has solutions $\mu = 1$ or $x = -y$. If $\mu = 1$, then $\lambda = z = 0$ (since $2x - 2(1)x = \lambda$), which simplifies the constraints to

$$x - y = 2 \tag{13}$$
$$x^2 + y^2 = 4. \tag{14}$$

(13) implies $x = 2 + y$, so $y^2 + (2 + y)^2 = 4 \implies 2y(y + 2) = 0$ so $y = 0, -2$. This corresponds to the points $(2, 0, 0)$ and $(0, -2, 0)$ from equation (13). Both of these points have a distance 2 from the origin. If $x = -y$ to satisfy equation (12), then the constraints become

$$2x + z = 2 \tag{15}$$
$$2x^2 = 4. \tag{16}$$

Again, (16) implies that $x = \pm\sqrt{2}$, so $y = \mp\sqrt{2}$, and $z = 2 \mp 2\sqrt{2}$. The next set of points we examine are then $\left(\sqrt{2}, -\sqrt{2}, 2 - 2\sqrt{2}\right)$ and $\left(-\sqrt{2}, \sqrt{2}, 2 + 2\sqrt{2}\right)$. The distance of these points are

$$\sqrt{2 + 2 + (2 \mp 2\sqrt{2})^2} = \sqrt{16 \pm 8\sqrt{2}} \tag{17}$$

which are both greater than 2, since $\sqrt{16} = 4 > 2$. Therefore the closest set of points to the origin in optimizing distance are $(2, 0, 0)$ and $(0, -2, 0)$.

Note: my method may be different from the method which you had learned in class - in this case, just compare your final answer with mine.

7. Let $H = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4,\ z \geq 0\}$ be the upper solid hemisphere of radius 2. Express the volume of H as an iterated double integral with integration order $dydx$.

Solution: We begin by identifying the integrand. In between the upper hemisphere of radius 2 and the xy-plane, we have z, so we integrate z between 0 and $z = +\sqrt{4 - x^2 - y^2}$ (upper hemisphere is $+$):

$$\int_0^{\sqrt{4-x^2-y^2}} dz = \sqrt{4 - x^2 - y^2}. \tag{1}$$

We now determine the integration bounds Since we are integrating only in x and y, the integrand must be a function of x and y only and this is what is given above by the z-integration in equation (1). The bounds of x and y are given when $z = 0$, the circle of radius 2 about the origin in the xy plane.

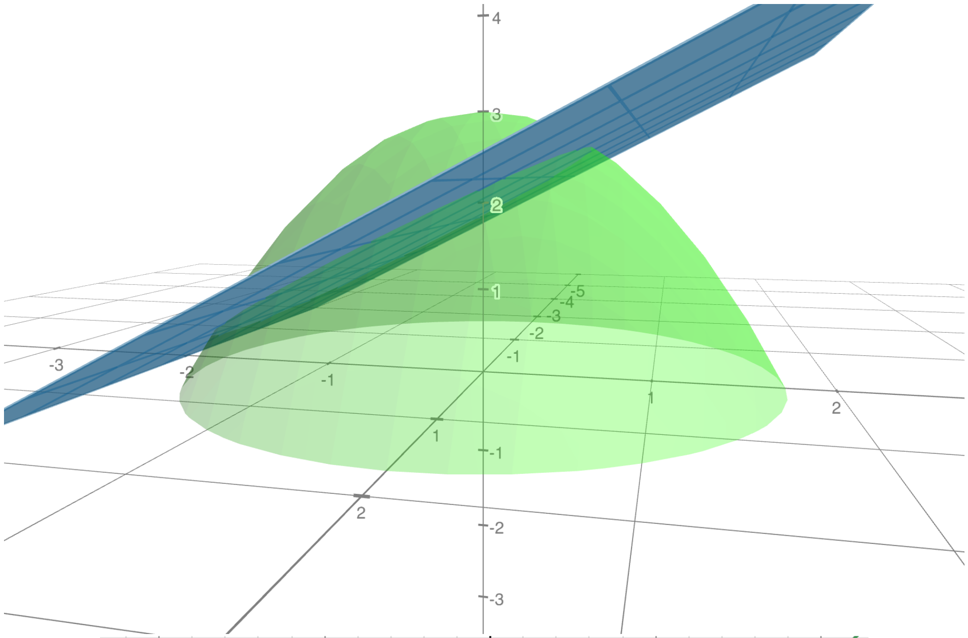
y must then integrate as a function of x, since we integrate y first, and these bounds are given by $y = \pm\sqrt{4 - x^2}$ to satisfy the $z = 0$ condition. We lastly observe that x must tend from $-2 \to 2$ to satisfy the $z = 0$ and $y = 0$ conditions. We therefore have

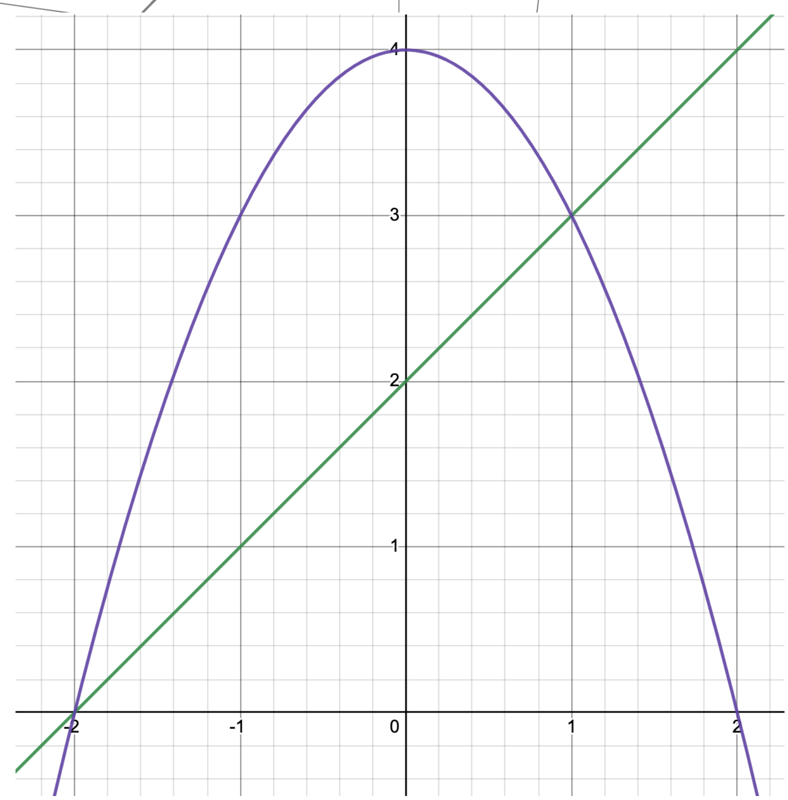
$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4 - x^2 - y^2}\, dydx \tag{2}$$

as an iterated double integral with integration order $dydx$.

8. Consider the solid S above $z = 0$, below $z = y + 2$, and below the paraboloid $z = 4 - x^2 - y^2$. Use <u>slices</u> to construct the volume of S in terms of iterated integrals with the order $dxdydz$. Explain the geometry of your process. Do not evaluate the integral.

Solution: The first suggested step would be to draw a diagram of the shape.





Above are attached images of the full volume and a slice at $x = 0$. At $x = 0$, we observe that the paraboloid and the plane intersect at two locations:

$$y + 2 = 4 - y^2 \implies y^2 + y - 2 = (y+2)(y-1) = 0 \tag{1}$$

which is $y = -2, y = 1$.

We begin by identifying the bounds of the integration.

Since we are integrating x first, we can find that $z = 4 - x^2 - y^2$ implies that x must range from $-\sqrt{4 - y^2 - z} \to \sqrt{4 - y^2 - z}$. It may appear confusing how the $y + 2$ plane is incorporated in this, and the simple answer is, it's not. This is because the plane is independent of x, only a function of y (and z). Therefore when we integrate x, we obtain functions of y and z from which we can impose the bounds on.

The next observation would to notice that y must range from -2 to 2, but under two different conditions. The first condition is when $-2 \leq y \leq 1$, the plane bounds the volume. The second is $1 \leq y \leq 2$, when the paraboloid bounds the volume. We can then integrate $y = z - 2 \to 1$, when the plane bounds, then add the integral for when $y = 1 \to \sqrt{4 - z}$ (remember that we've already integrated x, which is now a function of y and z). This produces a function of only z.

Lastly, the bounds for z are above the xy plane (so $z \geq 0$) and always less than 3, which is the most maximal point of intersection of the surfaces: $(1, 4 - (1)^2) = (1, 3)$. The integral in x is

$$g(y, z) = \int_{-\sqrt{4-y^2-z}}^{\sqrt{4-y^2-z}} dx \tag{2}$$

and in y,

$$h(z) = \int_{z-2}^{1} g(y, z)\, dy + \int_{1}^{\sqrt{4-z}} g(y, z)\, dy \tag{3}$$

and lastly in z,

$$\int_0^3 h(z)\, dz. \tag{4}$$

Putting (2), (3) and (4) altogether gives the volume inside the solid:

$$V = \int_0^3 \left[\int_{z-2}^{1} \int_{-\sqrt{4-y^2-z}}^{\sqrt{4-y^2-z}} dx dy + \int_{1}^{\sqrt{4-z}} \int_{-\sqrt{4-y^2-z}}^{\sqrt{4-y^2-z}} dx dy \right] dz \tag{5}$$

9. Consider the parallelogram $P = \{(x,y) \in \mathbb{R}^2 \,:\, 0 \le x - y \le 3, 1 \le x + y \le 3\}$ and its mass-density function $\delta : P \to \mathbb{R}$ given by $\delta(x,y) = (x+y)^2$. Use a change of variables to find the mass of P.

Solution: We can choose a change of variables to transform the parallelogram into a rectangle of side lengths 3 and 2. Let $u = x - y$ and $v = x + y$, so that P becomes

$$P = \{(u,v) \in \mathbb{R}^2 \,:\, 0 \le u \le 3, 1 \le v \le 3\}. \tag{1}$$

Under this substitution the mass density function also becomes $\delta(u,v) = v^2$. The mass of P is given by integration over the volume of the mass density function, $\text{mass}(P) = \iint_P \delta(u,v)\, dA$. The bounds of the integration are given by (1), but to determine the area element we must compute the stretch factor goverened by the Jacobian. Solving for u and v gives that $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(v-u)$, so we find

$$\begin{aligned}
dA &= |\det D(x,y)|\, dudv \qquad (2)\\
&= \det \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} dudv \qquad (3)\\
&= \det \begin{pmatrix} \dfrac{1}{2}\dfrac{\partial}{\partial u}(u+v) & \dfrac{1}{2}\dfrac{\partial}{\partial v}(u+v) \\ \\ \dfrac{1}{2}\dfrac{\partial}{\partial u}(v-u) & \dfrac{1}{2}\dfrac{\partial}{\partial v}(v-u) \end{pmatrix} dudv \qquad (4)\\
&= \det \begin{pmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \\ -\dfrac{1}{2} & \dfrac{1}{2} \end{pmatrix} dudv \qquad (5)\\
&= \left(\frac{1}{4} + \frac{1}{4}\right) dudv \qquad (6)\\
&= \frac{1}{2}\, dudv.
\end{aligned}$$

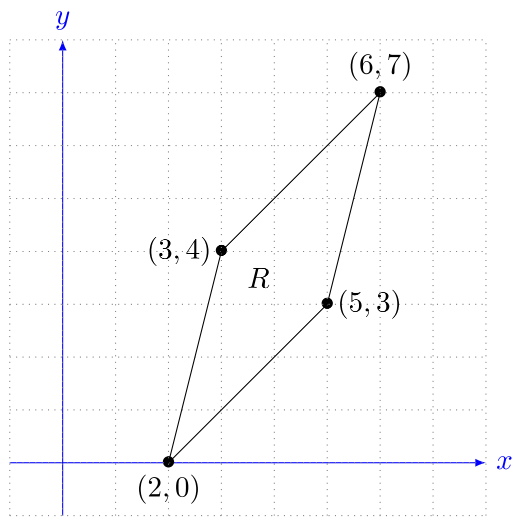
Thus the stretch factor is $\frac{1}{2}$. The mass is then

$$\begin{aligned}
\text{mass}(P) &= \iint_P \delta(u,v)\, dA \qquad (7)\\
&= \int_1^3 \int_0^3 v^2\, \frac{1}{2} dudv \qquad (8)\\
&= \frac{1}{2} \int_0^3 du \int_1^3 v^2\, dv \qquad (9)\\
&= \frac{1}{2} \cdot 3 \cdot \frac{1}{3} v^3 \Big|_1^3 \qquad (10)\\
&= \frac{1}{2}\,[27 - 1] \qquad (11)\\
&= 13. \qquad (14)
\end{aligned}$$

10. Evaluate $\displaystyle\iint_R (2x - \pi y)\, dA$ where R is the parallelogram with vertices $(2,0), (5,3), (6,7), (3,4)$.
Hint: derive a change of variables.

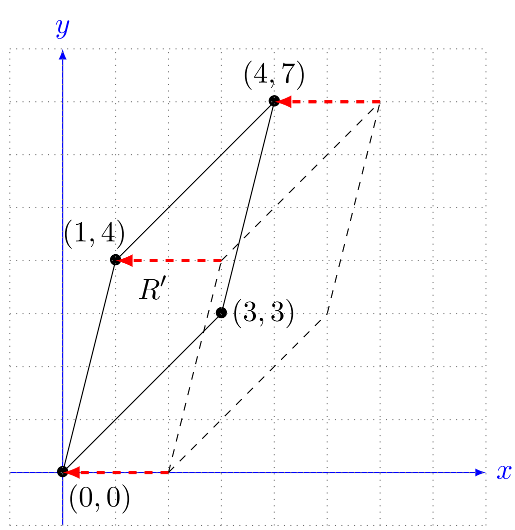
You are not expected to know linear algebra for MAT235, but I wanted to showcase this problem as it's important to understand how change of variables work in parallel to linear transformations. This problem is on the tougher side, as it requires some thinking and prerequisite knowledge from a course like MAT223.

Solution: The parallelogram R is shown below:



Let's label the vectors of the parallelogram $\mathbf{a} = (3,4)$ and $\mathbf{b} = (5,3)$ so that the area spanned by this parallelogram is $\mathbf{a} \times \mathbf{b}$. The goal is to determine some kind of linear transformation matrix which takes in vectors in R and maps them to the unit square spanned by $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, that is, $(1,0)$ and $(0,1)$.

We note that R is not centered at the origin, but at (2,0). Thus we can begin by translating R by having every $x \to x - 2$. This implies the first change of variables, $x' = x - 2$, $y' = y$ (the prime doesn't represent derivative, it just means that this is a secondary coordinate system), implying that R becomes



Since translations don't change the area or shape of R itself, the determinant of the Jacobian remains 1 so $dA' = dA$. The bounds of the integral are still unknown (which is ok!), but the integrand substitutes $x = x' + 2$, hence $2(x'+2) - \pi y'$ is the new integrand. We have

$$\iint_{R'} (2(x'+2) - \pi y')\, dA' \tag{1}$$

after the first translation.

Now that R' is fixed at the origin, we can find a linear transformation which takes in vectors $(1,0)$ and $(0,1)$ and maps them to $(3,3)$ and $(1,4)$ respectively (this is called a transformation matrix, https://en.wikipedia.org/wiki/Transformation_matrix). One can verify that the following matrix multiplication gives that

$$\begin{pmatrix} 3 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} 3 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \tag{3}$$

hence $M = \begin{pmatrix} 3 & 1 \\ 3 & 4 \end{pmatrix}$ is the matrix of the linear transformation: the matrix which takes in vectors in the unit square and maps them to R'. However, we want to find a linear transformation which takes vectors in R' and maps them to the unit square, which is equivalently the inverse of M. The inverse of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$A^{-1} = \frac{1}{\det A} \operatorname{Adj}(A) \tag{4}$$

$$= \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{5}$$

which, for M, is

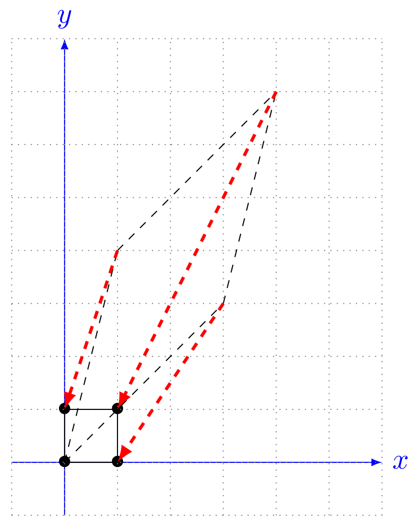
$$M^{-1} = \frac{1}{(3)(4) - (3)(1)} \begin{pmatrix} 4 & -1 \\ -3 & 3 \end{pmatrix} \tag{6}$$

$$= \frac{1}{9} \begin{pmatrix} 4 & -1 \\ -3 & 3 \end{pmatrix} \tag{7}$$

$$= \begin{pmatrix} 4/9 & -1/9 \\ -1/3 & 1/3 \end{pmatrix} \tag{8}$$

Therefore M^{-1} takes vectors in R' and maps them to the unit square, as desired.

How does this relate back to multivariable change of variables? Recall that the determinant of the Jacobian of a transformation $\det D(x, y)$ governs the scaling of unit area under the transformation. Equivalently, in linear algebra, $\det M^{-1}$ governs the stretching of unit area under a linear transformation to take R' to $[0,1] \times [0,1]$. Hence M^{-1} is the Jacobian of the transformation which takes points in R' to the unit square.



We therefore have

$$\begin{pmatrix} \dfrac{\partial u}{\partial x'} & \dfrac{\partial u}{\partial y'} \\ \\ \dfrac{\partial v}{\partial x'} & \dfrac{\partial v}{\partial y'} \end{pmatrix} = \begin{pmatrix} 4/9 & -1/9 \\ -1/3 & 1/3 \end{pmatrix} = M^{-1} \tag{9}$$

from which implies that, upon matching components and integrating,

$$u = \frac{1}{9}(4x' - y') \tag{10}$$

$$v = \frac{1}{3}(-x' + y') \tag{11}$$

which is the change of variables which takes in points $(x', y') \in R'$ and maps them to the unit square. One may verify that R' is the set

$$R' = \{(x', y') \in \mathbb{R}^2 \,:\, 0 \leq x' - y' \leq 3,\, 0 \leq 4x' - y' \leq 9\}. \tag{12}$$

In order to compute the initial integral, we must determine the bounds of the integral, the stretching factor, and the integrand. It is easy to see that, after our transformation to (u, v) coordinates, that u and v are both bound between 0 and 1 (it is the unit square), thus these are the integration bounds. The area element is

$$dA' = |\det M^{-1}|\, dudv \tag{13}$$

$$= \left| \frac{4}{9}\frac{1}{3} - \left(-\frac{1}{3}\right)\left(-\frac{1}{9}\right) \right| dudv \tag{14}$$

$$= \left| \frac{4-1}{27} \right| dudv \tag{15}$$

$$= \frac{1}{9}\, dudv. \tag{16}$$

The integrand can be determined by solving for x' and y', from which are implicitly given in (10) and (11). We find

$$9u = 4x' - y' \implies y' = 4x' - 9u \tag{17}$$

$$
\begin{aligned}
3v = -x' + y' &\implies 3v + x' = 4x' - 9u \implies x' = v + 3u && (18)\\
&\implies y' = 4v + 3u && (19)
\end{aligned}
$$

thus the integrand becomes

$$
\begin{aligned}
2x - \pi y &= 2(x' + 2) - \pi y' && (20)\\
&= 2(v + 3u) + 4 - \pi(4v + 3u) && (21)\\
&= 2v - 4\pi v + 6u - 3\pi u + 4 && (22)\\
&= (2 - 4\pi)v + (6 - 3\pi)u + 4 && (23)
\end{aligned}
$$

Lastly, combining the first translation, the determined bounds, equation (16) and (23), we can compute the integral:

$$
\begin{aligned}
\iint_R (2x - \pi y)\, dA &= \int_0^1 \int_0^1 \left[(2 - 4\pi)v + (6 - 3\pi)u + 4\right] \frac{1}{9}\, dudv && (24)\\
&= \frac{1}{9} \int_0^1 \left[(2 - 4\pi)uv + \frac{1}{2}(6 - 3\pi)u^2 + 4u\right] \Bigg|_0^1 dv && (25)\\
&= \frac{1}{9} \int_0^1 \left[(2 - 4\pi)v + \frac{1}{2}(6 - 3\pi) + 4\right] dv && (26)\\
&= \frac{1}{9} \left[\frac{1}{2}(2 - 4\pi)v^2 + \frac{1}{2}(6 - 3\pi)v + 4v\right] \Bigg|_0^1 && (27)\\
&= \frac{1}{9} \left[\frac{1}{2}(2 - 4\pi) + \frac{1}{2}(6 - 3\pi) + 4\right] && (28)\\
&= \frac{8}{9} - \frac{7\pi}{18} && (29)\\
&\approx -0.333. && (30)
\end{aligned}
$$

11. Find the extreme values of the function $g(x,y) = x^2 + y^2 - 2x + 3y + 5$ on the region $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$.

Solution: Begin by noting that D is a compact set. This implies we must both check the boundary ∂D and the interior D° for extrema.

On the interior, we take the gradient and evaluate critical points.

$$\nabla g(x,y) = \langle 2x - 2,\, 2y + 3\rangle = \mathbf{0} \tag{1}$$

which implies $\left(1, -\dfrac{3}{2}\right)$ is a critical point. Note that $(1)^2 + \left(\dfrac{3}{2}\right)^2 = \dfrac{13}{4} < 4$ so it is an interior critical point.

On the boundary, we use Lagrange multipliers. We set our constraint to be our boundary: $h(x,y) = x^2 + y^2 = 4$. Thus, using $\nabla g = \lambda \nabla h$, we have the system

$$x^2 + y^2 = 4 \tag{2}$$
$$2x - 2 = 2\lambda x \tag{3}$$
$$2y + 3 = 2\lambda y \tag{4}$$

Using (3), we have that $x(1-\lambda) = 1$ and using (4), $y(1-\lambda) = -\dfrac{3}{2}$. These imply that $y = -\dfrac{3}{2}x$ (we didn't explicity solve for λ here). Hence, using (2),

$$x^2 + \frac{9}{4}x^2 = 4 \implies x^2 = \frac{16}{13} \implies y^2 = \frac{36}{13}.$$

Hence there are 4 locations of critical points on the boundary.

At $(1, -1.5)$, $g(1, -1.5) = 1.75$. We also have

$$g(1.109..., 1.664...) \approx 11.77 \tag{5}$$
$$g(-1.109..., 1.664...) \approx 16.21 \tag{6}$$
$$g(1.109..., -1.664...) \approx 1.789 \tag{7}$$
$$g(-1.109..., -1.664...) \approx 6.226 \tag{8}$$

hence g has a global minimum on the set D at $(1, -1.5)$ and a global maximum on D at $\left(-\dfrac{4}{\sqrt{13}}, \dfrac{6}{\sqrt{13}}\right)$ on the boundary.

12. Let T be the part of the solid cylinder $x^2 + y^2 \leq 6$ which lies above the xy-plane and below the sphere $x^2 + y^2 + z^2 = 16$. Evaluate the triple integral

$$\iiint_T f(x, y, z)\, dV$$

where $f(x, y, z) = x + z$.

Solution: Use cylindrical coordinates by the symmetry of the problem. Let $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$ so that $dV = r\,drd\theta dz$. In these coordinates, we integrate over $r^2 \leq 6$, under the sphere $z = \sqrt{16 - r^2}$ (above xy plane, we take + root).

The integrand becomes $f(r, \theta, z) = r\cos\theta + z$. Since the height of our volume is goverened as a function of r, we integrate z before r. Thus we have

$$\int_0^{2\pi} \int_0^{\sqrt{6}} \int_0^{\sqrt{16-r^2}} [r\cos\theta + z] r\, dzdrd\theta. \tag{1}$$

Upon integration over θ from $0 \to 2\pi$, the $r\cos\theta$ integral vanishes since we integrate $\cos\theta$ over it's period, which is 0. Taking out the integrals, we have

$$\begin{aligned}
\int_0^{2\pi} \int_0^{\sqrt{6}} \int_0^{\sqrt{16-r^2}} [r\cos\theta + z] r\, dzdrd\theta &= 2\pi \int_0^{\sqrt{6}} \int_0^{\sqrt{16-r^2}} zr\, dzdr & (2)\\
&= 2\pi \int_0^{\sqrt{6}} \frac{1}{2}\left[z^2\right]_0^{\sqrt{16-r^2}} r & (3)\\
&= \pi \int_0^{\sqrt{6}} r(16 - r^2)\, dr & (4)\\
&= \pi \left[8r^2 - \frac{1}{4} r^4\right]_0^{\sqrt{6}} & (5)\\
&= \pi \left[(8)(6) + \frac{1}{4}(36)\right] & (6)\\
&= 57\pi. & (7)
\end{aligned}$$