MAT235 Tutorial 11

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This week: parametrizations and line integrals. Refer 17.1, 18.2 Hughes-Hallett.

Parametrizations of lines/functions: Suppose we want to parametrize a line in $\mathbb{R}^n$ which passes through two points $\mathbf{a}$ to $\mathbf{b}$. This mapping becomes a curve $\ell : [0,1] \to \mathbb{R}^n$ given by $\ell(t) = \mathbf{a}(1-t) + \mathbf{b}t$, where t is the parameter. For a (surjective) function $f : [p,q] \to \mathbb{R}$ (for lines in $\mathbb{R}^2$), the parametrization is simply given by $y = f(x)$. That is, over the interval $[p,q]$, $c : [p,q] \to \mathbb{R}^2$ by $c(t) = \langle t, f(t) \rangle$.
In $\mathbb{R}^3$, we also examine surface parametrizations. The general rule is, for a parametrization $g : \mathbb{R}^n \to \mathbb{R}^m$, that $m > n$. This means that whatever object we are parametrizing must have a degree of dimensionality less than that of the space where it is located. For instance, in our example above, c maps a 1-dimensional surface (a curve) into 2-dimensional space and $2 > 1$. A surface parametrization in $\mathbb{R}^3$ would look like $\sigma : [p,q] \times [r,s] \to \mathbb{R}^3$ where $\sigma(s,t) = \langle\ ,\ ,\ \rangle$ $(3 > 2)$. A curve/line in $\mathbb{R}^3$ is similar: $c(t) = \langle\ ,\ ,\ \rangle$ $(3 > 1)$.

Problems:

1. Let C be the curve consisting of the arc C_1 of the graph $y = x^2 - x$ from $(0,0)$ to $(1,0)$, followed by the line segment C_2 from $(1,0)$ to $(0,0)$.
   (a) Find parametrizations for C_1 and C_2.
   (b) Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field given by $\mathbf{F}(x,y) = \langle -y, x \rangle$. Calculate $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

2. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field given in component form by $\mathbf{F}(x,y) = (x+2y)\hat{\mathbf{i}} + (x-y)\hat{\mathbf{j}}$. Let C_1 be the line segment from $(0,0)$ to $(2,2)$, and let C_2 be the curve parametrized by $\mathbf{r}(t) = \left\langle t, \dfrac{t^2}{2} \right\rangle$ where $t \in [0,2]$.
   (a) Find $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.
   (b) Are the line integrals equal? What does that suggest about $\mathbf{F}$?
   (c) Repeat the above for $\mathbf{G}(x,y) = (x+2y)\hat{\mathbf{i}} + (2x-y)\hat{\mathbf{j}}$. What changed? Is $\mathbf{G}$ conservative?