MAT235 Tutorial 12

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This week: the curl test and Green's theorem. Refer 18.3 - 18.4 Hughes-Hallett.

Curl Test: Let $f : \mathcal{U} \to \mathbb{R}$ be a real-valued function which is C^2, at least, on its domain $\mathcal{U} \subseteq \mathbb{R}^3$. Then $\nabla \times (\nabla f) = \mathbf{0}$.
Proof: The gradient of f is

$$\nabla f(x, y, z) = \langle \partial_x f, \, \partial_y f, \, \partial_z f \rangle \tag{1}$$

Since f is a C^2 function, then $(\nabla f) : \mathcal{U} \to \mathbb{R}^3$ is also differentiable, so we may take the curl to determine the counter-clockwise rotational component:

$$\begin{aligned}
\nabla \times (\nabla f) &= \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ \partial_x f & \partial_y f & \partial_z f \end{pmatrix} && (2) \\
&= \hat{\mathbf{i}}(\partial_y \partial_z f - \partial_z \partial_y f) + \mathbf{j}(\partial_z \partial_x f - \partial_x \partial_z f) + \hat{\mathbf{k}}(\partial_x \partial_y f - \partial_y \partial_x f) && (3) \\
&= \hat{\mathbf{i}}(\partial_y \partial_z f - \partial_y \partial_z f) + \mathbf{j}(\partial_z \partial_x f - \partial_z \partial_x f) + \hat{\mathbf{k}}(\partial_x \partial_y f - \partial_x \partial_y f) && (4) \\
&= \mathbf{0} && (5)
\end{aligned}$$

where (4) follows from Clairaut's theorem. ■

Path-Independence: Let $\mathbf{G} : \mathcal{U} \to \mathbb{R}^3$ be a vector field which is C^1, at least, on its domain $\mathcal{U} \subseteq \mathbb{R}^3$ where $\mathcal{U}$ is path-connected. If $\nabla \times \mathbf{G} = \mathbf{0}$, then $\mathbf{G}$ is path-independent for any line integral over any curve $C \subseteq \mathcal{U}$.
Proof: Let $\mathbf{r} : [a, b] \to \mathcal{U}$ be the parametrization of C. Assume $\nabla \times \mathbf{G} = \mathbf{0}$. Then, by the curl test, there exists a C^2 function f such that $\mathbf{G} = \nabla f$. By the fundamental theorem of line integrals, one has

$$\begin{aligned}
\int_{\mathbf{r}(a)}^{\mathbf{r}(b)} \mathbf{G} \cdot d\mathbf{r} &= \int_{\mathbf{r}(a)}^{\mathbf{r}(b)} (\nabla f) \cdot d\mathbf{r} && (6) \\
&= f(\mathbf{r}(a)) - f(\mathbf{r}(b)) && (7)
\end{aligned}$$

and therefore any line integral of $\mathbf{G}$ would yield the same value, so $\mathbf{G}$ is a conservative vector field. ■

Green's Theorem: Let C be a positively oriented, piecewise smooth, simple closed curve in $\mathbb{R}^2$. Let D be a set such that $C = \partial D$ (D is the interior enclosed by C). Let $\mathbf{F}(x, y) = P(x, y)\hat{\mathbf{i}} + Q(x, y)\hat{\mathbf{j}}$ be a C^1 vector field, at least, on an open region containing D (P and Q are both $\mathbb{R}^2 \to \mathbb{R}$ and are C^1 over D). Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dA. \tag{8}$$

Note on Notation: For a gradient operation, $\nabla f \equiv \text{grad}(f)$. For a curl operation, $\nabla \times \mathbf{F} \equiv \text{curl}(\mathbf{F})$. For a divergence operation, $\nabla \cdot \mathbf{F} \equiv \text{div}(\mathbf{F})$.

Problems:

1. For each of the following vector fields on $\mathbb{R}^3$, use the curl test to determine if the vector field is a gradient vector field. If so, find the function f such that $\mathbf{F} = \nabla f$.

   (a) $\mathbf{F}(x, y, z) = z\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$

   (b) $\mathbf{F}(x, y, z) = \langle yz, xz, xy\rangle$

2. Evaluate the line integral $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy^2\,\hat{\mathbf{i}} + \cos x\,\hat{\mathbf{j}}$, and C is the boundary of the region enclosed by the curves $y = |x|$ and $y = 1$.