MAT235 Tutorial 13

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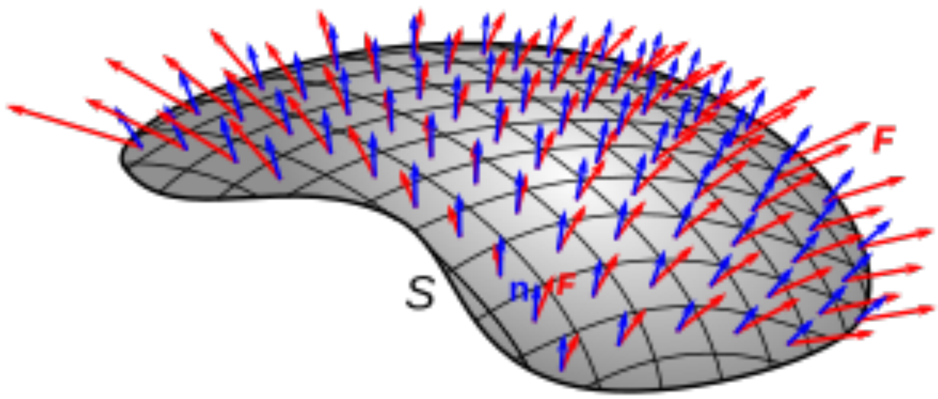
This week: surface integrals. Refer 19.2, 21.1, 21.3 Hughes-Hallett.

Surface Orientations and Normal Vectors: Recall from your computations of tangents planes at a point on a surface that a planes orientation is dictated by its normal vector, pointing orthogonal to the plane. It is the same for infinitessimal area elements $d\mathbf{A}$ while computing surface integrals. Each divison of the surface contains its on area element and we integrate over the entire surface via a parametrization. By convention, all normal vectors point **outward** from the surface.



[Wikipedia] Normal vector orientation (blue) regarding the flux of a vector field (red) through a surface S.

The scaling of an area is proportional to the length of it's normal vector. Thus, for a parametrization of an area $\mathbf{r} : [a,b] \times [c,d] \to \mathbb{R}^3$ given by $\mathbf{r}(u,v)$, the normal vector is given by the cross product $\mathbf{r}_u \times \mathbf{r}_v$. The length of this vector is $|\mathbf{r}_u \times \mathbf{r}_v|$, which is what we integrate over. Hence, for S,

$$\iint_S dA = \iint_{[a,b]\times[c,d]} |\mathbf{r}_u \times \mathbf{r}_v| dA_{(u,v)}. \tag{1}$$

Introduction to Stoke's Theorem: Recall last week Green's theorem: that the net curl of a vector field over a set in $\mathbb{R}^2$ is equal to the closed line integral of the field around the boundary of said set. This is actually just a reduction of Stoke's theorem, which says that for any surface S in $\mathbb{R}^3$, the net curl of a field $\mathbf{F}$ defined, at least, on S, is equivalent to the line integral around ∂S (the boundary) for *any* orientation of S, not just in $\mathbb{R}^2$. This also means that S is not restricted to a plane, but any surface.

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dA \tag{2}$$

where $\hat{\mathbf{n}}$ is the normal vector of the surface as a function of the parametrization variables.

Problems:

1. Find the surface area of the solid bounded by the planes $y + z = 6$, $z = 0$ and the cylinder $x^2 + y^2 = 4$.

2. Let S be the unit circle centred at the origin. Use a parametrization to find a formula for the surface area of the restriction of S which lies above the $z = \frac{1}{2}$ plane.