MAT235 Tutorial 2

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This week: multivariable limits and the Squeeze theorem. Refer 12.6 Hughes-Hallett.

Squeeze Theorem (Single-Variable): Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and $h : \mathbb{R} \to \mathbb{R}$ be functions. Let $a \in \mathbb{R}$ be a point. Suppose that for all $x \in \mathbb{R}$ close to a, but not equal to a, that $f(x) \leq g(x) \leq h(x)$.
If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

Although this is a simplification (some functions may only be defined on some intervals on $\mathbb{R}$, so we're making an assumption here - this will need to be fixed when we move to $\mathbb{R}^2$), it can be generalized to multivariable limits as well.

Squeeze Theorem (Multi-Variable): Let $A \subseteq \mathbb{R}^2$ be path-connected. Let $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$, and $h : A \to \mathbb{R}$ be functions defined, at least, on A. Let $(a, b) \in A$ be a point. Suppose that for all $(x, y) \in A$ close to (a, b), but not equal to (a, b), that $f(x, y) \leq g(x, y) \leq h(x, y)$.
If $\lim_{(x,y)\to(a,b)} f(x, y) = L$ and $\lim_{(x,y)\to(a,b)} h(x, y) = L$, then $\lim_{(x,y)\to(a,b)} g(x, y) = L$.

No need to be confused by the types of sets, since this is just a formal definition (you won't be tested on these). What's important is that if $f(x, y) \leq g(x, y) \leq h(x, y)$ and if $\lim_{(x,y)\to(a,b)} f(x, y) = L$ and $\lim_{(x,y)\to(a,b)} h(x, y) = L$, then $\lim_{(x,y)\to(a,b)} g(x, y) = L$. Let's implement this into an example.

Example: Consider the function $f : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$ defined by $f(x, y) = \dfrac{x^2y^2}{x^2 + y^2}$. Note that f is undefined at the origin. Let's evaluate $\lim_{(x,y)\to(0,0)} f(x, y)$.

First observe that f is strictly positive, $\dfrac{x^2y^2}{x^2 + y^2} > 0$ for all $(x, y) \in \mathbb{R} \setminus \{(0, 0)\}$. This implies that $f(x, y)$ is bounded below by the $z = 0$ plane (the xy plane). To determine the upper bound, we should examine the relationship between then numerator and denominator in $f(x, y)$. If $y^2 \leq x^2 + y^2$, then $\dfrac{y^2}{x^2 + y^2} \leq 1$. Multiplying both sides by x^2, we find that $\dfrac{x^2y^2}{x^2 + y^2} = f(x, y) \leq x^2$. Thus $0 \leq f(x, y) \leq x^2$. As we approach $(0, 0)$, then, $x^2 \to 0$, so $f(x, y) \to 0$. Therefore, by the Squeeze theorem, $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.

You may usually find that determining the bounds on the function requires some out-of-the-box thinking, but the general procedure involves determining upper and lower bounds. One useful tool is examining the components of the function and analyzing each of the individual bounds to attempt to derive some relationship between them. For instance, considering functions that you already know are bounded like $\sin(x), \cos(x), \tan(x)$, simplifies the problem drastically as we will see.

Two-Path Test: Consider a real-valued function f defined on some set $U \subseteq \mathbb{R}^2$. Let $(a,b) \in U$. If the limit along one parametrized path of f is L, and the limit along another parametrized path of f is M, and $L \neq M$, then the limit does not exist.

More formally (if you require it), let $U \subseteq \mathbb{R}^2$ be path-connected and $f : U \to \mathbb{R}$. Let $\gamma_1 : [p,q] \to \mathbb{R}^2$, $\gamma_2 : [r,s] \to \mathbb{R}^2$ be parametrizations such that $\operatorname{im}(\gamma_1), \operatorname{im}(\gamma_2) \subseteq U$. Let $\alpha \in [p,q]$, $\beta \in [r,s]$ be points where $\lim\limits_{t\to\alpha} \gamma_1(t) = \lim\limits_{t\to\beta} \gamma_2(t) = (a,b)$. If $\lim\limits_{t\to\alpha} f(\gamma_1(t)) = L$ and $\lim\limits_{t\to\beta} f(\gamma_2(t)) = M$ where $L \neq M$, then the limit $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example: Consider the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^6}{xy^3}$. Show that the limit does not exist using the two-path test.
Consider the parametrization $x = y^3$, or $\gamma_1(y) = \langle y^3, y\rangle$. Then, as $(x,y) \to (0,0)$, $\gamma_1(y) \to (0,0)$ as $y \to 0$. This implies that

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^2 - y^6}{xy^3} &= \lim_{y\to 0} \frac{y^6 - y^6}{y^3 y^3} && (1)\\
&= \lim_{y\to 0} \frac{0}{y^3 y^3} && (2)\\
&= 0. && (3)
\end{aligned}$$

Now consider the parametrization $x = y$, or $\gamma_2(y) = \langle y, y\rangle$. Then, as $(x,y) \to (0,0)$, $\gamma_2(y) \to (0,0)$ as $y \to 0$. Hence

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^2 - y^6}{xy^3} &= \lim_{y\to 0} \frac{y^2 - y^6}{y^4} && (4)\\
&= \lim_{y\to 0} \left[\frac{1}{y^2} - y^2\right] && (5)\\
&= \infty. && (6)
\end{aligned}$$

Since the first limit converges along one path, and the second diverges to $+\infty$ along another path, by the Two-Path Test, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^6}{xy^3}$ does not exist.

Problems (solutions posted on MAT235 Quercus; we'll take them up in tutorial):

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} \tag{7}$$

   Does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist? If so, what is it's value?

2. Let

$$g(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} \tag{8}$$

   Find, if they exist, $\lim\limits_{(x,y)\to(0,0)} g(x,y)$ and $\lim\limits_{(x,y)\to(1,1)} g(x,y)$.