MAT235 Tutorial 3

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This week: standard form of planes and normal vectors; dot product; cross product; adding vectors. Refer 13.2-13.3 Hughes-Hallett.

Planes and Normal Vectors: The standard form of a plane in $\mathbb{R}^3$ is given as

$$Ax + By + Cz + D = 0, \tag{1}$$

where A, B, C, D are real numbers. $-A/C, -B/C$, are the slopes of the plane along the x, y directions, respectively. You can see this by slicing the equation of the plane along $y = 0$, $x = 0$, (these slices yield the equation of a line, of the form $z = -\frac{A}{C}x - \frac{D}{C}$, $z = -\frac{B}{C} - \frac{D}{C}$, with a slope and $-\frac{D}{C}$ as the z-intercept). A, B, and C all imply the orientation of the plane, which can be represented as a normal vector: a vector $\mathbf{n}$ which is perpendicular (pointing outwards) to the plane at every point. This is simply given as

$$\mathbf{n} = \langle A, B, C \rangle. \tag{2}$$

I will leave it as an exercise for you to understand how (1) and (2) are related (recall the slope of a line is rise/run).

Dot Product: We can compare orientations of different planes using the dot product. For two vectors
$\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, their dot product is defined as a scalar projection of $\mathbf{u}$ onto $\mathbf{v}$, that is, the magnitude of the "shadow" of $\mathbf{u}$ on $\mathbf{v}$. It is given algebraically and geometrically:

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 \tag{3}$$

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta \tag{4}$$

and θ is the angle between the vectors. Hence planes are parallel if $\theta = 0$, and planes are perpendicular if $\theta = \frac{\pi}{2}$. Recall that the magnitude of a vector is $|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$ (see Pythagoras' theorem).

Cross Product: There is an equivalent vector projection of $\mathbf{u}$ and $\mathbf{v}$, called the cross product. Instead of yielding a scalar quantity, a vector quantity $\mathbf{w}$ is produced which is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. They may also be represented as algebraic and geometric formulas:

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = \det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} \tag{5}$$

$$= \langle u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1 \rangle \tag{6}$$

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = |\mathbf{u}||\mathbf{v}| \sin\theta\, \hat{\mathbf{s}} \tag{7}$$

where $\hat{\mathbf{s}}$ is the unit vector in the direction of $\mathbf{u} \times \mathbf{v}$, defined as $\hat{\mathbf{s}} = \frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u}||\mathbf{v}| \sin\theta}$. One convenient way to memorize the algebraic calculation is by using a determinant of a 3×3 matrix (if you haven't seen this yet that's ok, we'll look at it later, but you will need to know this especially for vector calculus). (7) implies that $\mathbf{w}$ is a maximum in magnitude when $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ and a minimum when $\theta = 0, \pi, \ldots$ (in fact, the value is 0 - and the unit vector is not defined, so we denote this magnitude as the zero vector: $\mathbf{0}$).

Sums of Vectors: Geometrically, when we add vectors, they are added tip-to-tail. The resulting addition/difference is called the "resultant" vector. Subtracting vectors implies adding the negative. Using $\mathbf{u}$ and $\mathbf{v}$ again, we compute their algebraic sum as

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1,\, u_2 + v_2,\, u_3 + v_3 \rangle. \tag{8}$$

which is just the sum of each of the respective components.

Problems (solutions posted on MAT235 Quercus; we'll take them up in tutorial):

1. Which of the following planes are parallel of each other? Which are perpendicular?
   (a) $2x + 3y - 5z = 4$
   (b) $2x - 3y - z = 3$
   (c) $4x + 6y - 2z = 0$
   (d) $z = 4 - 2x - 3y$
   (e) $z = 4 + 2x + 3y$
   (f) $x + y + z = 1$
2. True or False: the triangle T in $\mathbb{R}^3$ with vertices $(1, 1, 0)$, $(0, 1, 0)$, and $(0, 1, 1)$ is a right-angled triangle. Justify your answer.
3. Three ropes are attached to a metal ring. Three people pull each of the ropes with a different force in a different direction. Person 1 pulls the rope 300N due east, while person 2 pulls the second rope due south with a force of 400N.
   (a) In what direction, and with what force (give components), must person 3 pull the third rope so that the metal ring remains stationary.
   (b) What is the magnitude of the required force?