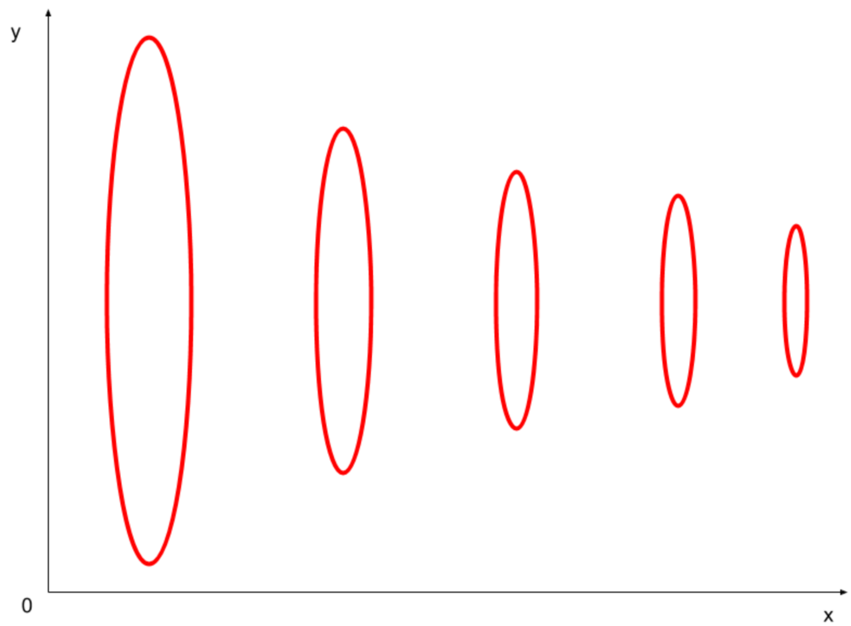
MAT235 Tutorial 4: Solutions to Additional Problems

TA: Jace Alloway (j.alloway@mail.utoronto.ca)
Office Hours: M14-16 PG101

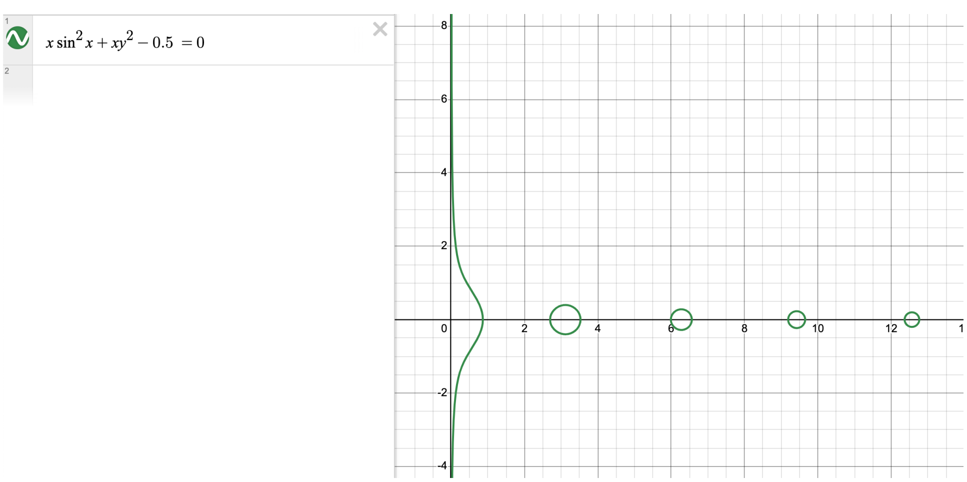
5. Attempt to write a function whose contours at $z = 0$ are similar to



Hint: I would suggest doing this in Desmos with an implicit function. There is no exact answer. You may place conditions on the domain of the function. The goal of this problem is to identify the relationships between variables, transformation operations, and implicit functions with contours.

Solution: The first goal is to write an implicit function which generates loops (or 'spikes' in $\mathbb{R}^3$), as shown above. I began with

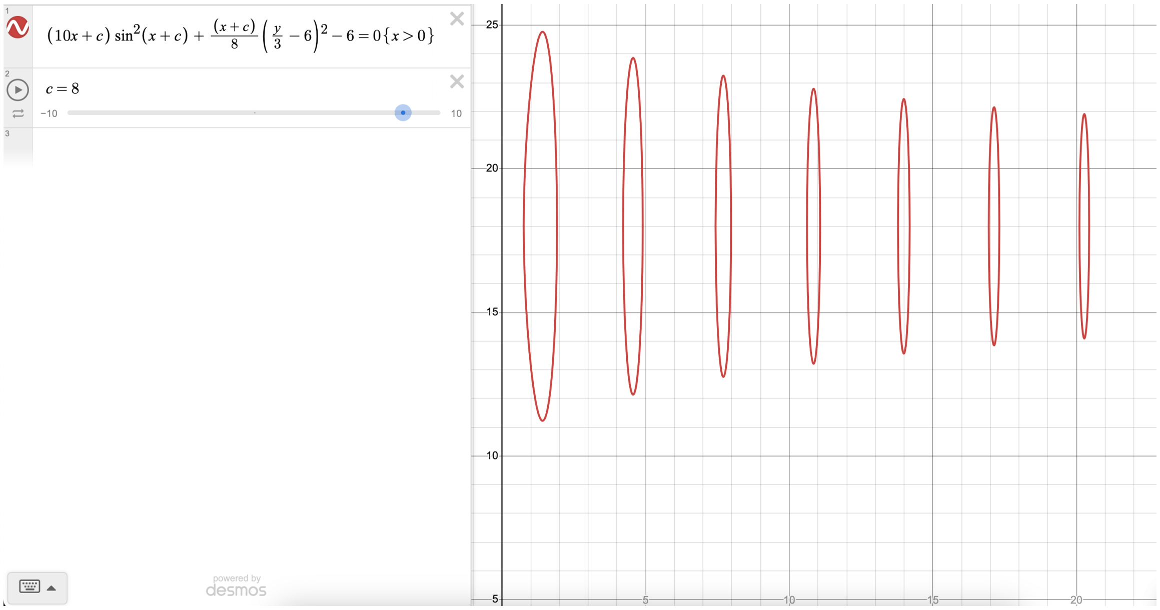
$$x\sin^2 x + xy^2 - 1 = 0. \tag{5.1}$$



Once implicit loops are achieved, one can perform scaling and transformations to adjust the size of the loops to form ellipses. I found that

$$(10x + c)\sin^2(x + c) + \frac{x + c}{8}\left(\frac{y}{3} - 6\right)^2 - 6 = 0 \tag{5.2}$$

for $c = 8$ worked best, for $x > 0$ and $y > 0$.



6. Let P be a plane containing the points $(4,-3,1), (-3,-1,1)$ and $(4,-2,8)$. Let L be a line parametrized by

$$\mathbf{r}(t) = \langle 10, 9, 5\rangle + t\langle 5, 3, 1\rangle.$$

Determine whether P and L intersect. If they do, find the point of intersection.

Solution: We will begin by finding the standard form equation for the plane P. Two vectors which span P are

$$\begin{aligned}
\mathbf{v}_1 &= (4,-3,1) - (-3,-1,1) && (6.1)\\
&= \langle 7, -2, 0\rangle && (6.2)\\
\mathbf{v}_2 &= (4,-2,8) - (-3,-1,1) && (6.3)\\
&= \langle 7, -1, 7\rangle. && (6.3)
\end{aligned}$$

Taking the cross product of $\mathbf{v}_1$ and $\mathbf{v}_2$ yields a vector orthogonal to P (the normal vector). That is

$$\begin{aligned}
\mathbf{n} &= \det\begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 7 & -2 & 0 \\ 7 & -1 & 7 \end{pmatrix} && (6.4)\\
&= \langle (-2)(7) - (-1)(0),\ (7)(0) - (7)(7),\ (7)(-1) - (7)(-2)\rangle && (6.5)\\
&= \langle -14,\ -49,\ 7\rangle && (6.6)\\
&= 7\,\langle -2, -7, 1\rangle. && (6.7)
\end{aligned}$$

Hence

$$-2x - 7y + z + D = 0 \qquad (6.8)$$

where D is a translation determined by a point in the plane. To find D, we can plug in the point $(4,-3,1)$:

$$\begin{aligned}
D &= 2(4) + 7(-3) - (1) && (6.9)\\
&= -14. && (6.10)
\end{aligned}$$

Therefore the standard form equation of P is

$$-2x - 7y + z - 14 = 0. \qquad (6.11)$$

We next determine whether or not L and P are parallel or orthogonal. Let $\mathbf{u} = \langle 5, 3, 1\rangle$ be the direction vector of L. If the dot product between $\mathbf{u}$ and $\mathbf{n}$ is zero, then the plane and line are parallel to each other and will never intersect. Otherwise, since L and P are infinitely long, there exists a point of intersection. We find

$$\mathbf{u}\cdot\mathbf{n} = (5)(-2) + (3)(-7) + (1)(1) = -30 \qquad (6.12)$$

which is non-zero. We can now determine the intersection point. Obtaining

$$\begin{aligned}
x(t) &= 10 + 5t && (6.13)\\
y(t) &= 9 + 3t && (6.14)\\
z(t) &= 5 + t && (6.15)
\end{aligned}$$

from the parametrization of L, we can plug these parametrizations into (6.11) and solve for t, which will give the parametrization value of the intersection location.

$$
\begin{aligned}
0 &= -2(10+5t) - 7(9+3t) + (5+t) - 14 && (6.16)\\
&= -20 - 63 + 5 - 14 - 10t - 21t + t && (6.17)\\
&= -92 - 30t && (6.18)\\
\implies t &= -\frac{92}{30} = -\frac{46}{15}. && (6.19)
\end{aligned}
$$

The corresponding point of intersection can be found by evaluating $\mathbf{r}(-32/25)$:

$$
\begin{aligned}
\mathbf{r}(-46/15) &= \langle 10, 9, 5\rangle - \frac{46}{15}\langle 5, 3, 1\rangle && (6.20)\\
&= \left(-\frac{16}{3}, -\frac{1}{5}, \frac{29}{15}\right). && (6.21)
\end{aligned}
$$

7. Evaluate the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{x^5+y^3}.$$

Solution: Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined via

$$f(x,y) = \frac{x^2y^3}{x^5+y^3}. \tag{7.1}$$

The first key observation is that f is undefined for all of $x^5 = -y^3$; a crevasse in the xy plane. Since the limit approaches the origin, and f is undefined at the origin, we cannot simply evaluating $f(0,0)$ directly. We further note that f is unbounded above and below, so we cannot apply the Squeeze theorem. The denominator is not strictly non-negative or non-positive, and the numerator has the unbounded term y^3.

The next guess would be to show that the limit does not exist. We can do this by applying the two-path test. The first parametrization we can choose is $x = y$ (since $x = 0$ or $y = 0$ are both zero), so let

$$\gamma_1(t) = \langle t, t\rangle \tag{7.2}$$

for $t\to 0$. We find that

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{t\to 0} f(\gamma_1(t)) && (7.3)\\
&= \lim_{t\to 0} \frac{t^5}{t^5+t^3} && (7.4)\\
&= \lim_{t\to 0} \frac{t^2}{t^2+1} && (7.5)\\
&= 0 && (7.6)
\end{aligned}$$

which is the same as if $x = 0$ or $y = 0$. The next parametrization would need to be something much more unique, and one path we can choose is any path close to the crevasse $x^5 = -y^3$ (there are more instabilities near asymptotes). Let us therefore choose the nearby path $x^5 - x^\alpha = -y^3$ for a constant α. The parametrization is

$$\gamma_2^\alpha(t) = \langle t, (t^\alpha - t^5)^{1/3}\rangle \tag{7.7}$$

which approaches 0 as $t\to 0$. Thus

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{t\to 0} f(\gamma_2^\alpha(t)) && (7.8)\\
&= \lim_{t\to 0} \frac{t^2(t^\alpha - t^5)^{3/3}}{t^5 + (t^\alpha - t^5)^{3/3}} && (7.9)\\
&= \lim_{t\to 0} \frac{t^{2+\alpha} - t^7}{t^\alpha}. && (7.10)
\end{aligned}$$

We can now see that if $\alpha < 7$, the limit will still evaluate to zero. However, if $\alpha = 7$, the limit will actually be -1:

$$\begin{aligned}
\lim_{t\to 0} f(\gamma_2^7(t)) &= \lim_{t\to 0} \frac{t^9 - t^7}{t^7} && (7.11)\\
&= \lim_{t\to 0} t^2 - 1 && (7.12)
\end{aligned}$$

$$= -1 \tag{7.13}$$

and if $\alpha > 7$ (I will just pick $\alpha = 9$, but any value over 7 will yield the same result), the limit diverges completely:

$$\lim_{t\to 0} f(\boldsymbol{\gamma}_2^9(t)) = \lim_{t\to 0} \frac{t^{11} - t^7}{t^9} \tag{7.14}$$

$$= \lim_{t\to 0} t^2 - \frac{1}{t^2} \tag{7.15}$$

$$= -\infty. \tag{7.16}$$

You can also check α for fractions:

$$\lim_{t\to 0^+} f(\boldsymbol{\gamma}_2^{7.2}(t)) = -\infty \tag{7.17}$$

$$\lim_{t\to 0^-} f(\boldsymbol{\gamma}_2^{7.2}(t)) = +\infty \tag{7.18}$$

Hence, by the two-path test, the limit doesn't exist.

8. Evaluate the limit

$$\lim_{(x,y)\to(0,0)} \frac{y^3 e^{-y^2}\sin(xy)}{xy}.$$

Solution: Let $g:\mathbb{R}^2 \to \mathbb{R}$ be given by

$$g(x,y) = \frac{y^3 e^{-y^2}\sin(xy)}{xy}. \tag{8.1}$$

To evaluate

$$\lim_{(x,y)\to(0,0)} g(x,y), \tag{8.2}$$

we cannot simply evaluate $g(0,0)$, since g is undefined at the origin. We can instead show that the limit exists using the Squeeze theorem. Let us first write g as a seperable function (product between x and y terms):

$$g(x,y) = y^2 e^{-y^2} \cdot \frac{\sin(xy)}{x}. \tag{8.3}$$

First recall that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$. Furthermore, note that, as we approach $x = 0$, we can apply the single-variable Taylor approximation of $\sin x$ at $x = 0$:

$$\sin x \approx \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \tag{8.4}$$

Since we are approaching 0 in the limit, higher terms aren't necessary in the expansion. We can then just keep the first term. In the $\sin(xy)$ of (8.3), then, we can just write

$$\sin(xy) \approx xy \tag{8.5}$$

hence

$$\frac{\sin(xy)}{x} \approx y \tag{8.6}$$

You will later see that multivariable Taylor expansions aren't the same as single variable ones. This formula is not general, even at first order. Using (8.6), we can approximate the limit as

$$\lim_{(x,y)\to(0,0)} y^3 e^{-y^2}. \tag{8.7}$$

The next observation would be to note that the Gaussian is bounded:

$$0 \le e^{-y^2} \le 1. \tag{8.8}$$

hence

$$0 \le y^3 e^{-y^2} \le y^3 \qquad \{y \ge 0\} \tag{8.9}$$

is true for <u>positive</u> values of y (this doesn't make sense if y is negative). One can apply the same idea for the left-hand side:

$$y^3 \le y^3 e^{-y^2} \le 0 \qquad \{y \le 0\} \tag{8.10}$$

for <u>negative</u> values of y. In either case, the function is bounded above and below:

$$\lim_{y\to 0^-} y^3 \le \lim_{y\to 0^-} y^3 e^{-y^2} \le 0 \tag{8.11}$$

$$\implies \lim_{y\to 0^-} y^3 e^{-y^2} = 0 \tag{8.12}$$

$$0 \le \lim_{y\to 0^+} y^3 e^{-y^2} \le \lim_{y\to 0^+} y^3 \tag{8.13}$$

$$\implies \lim_{y\to 0^+} y^3 e^{-y^2} = 0 \tag{8.14}$$

where I have omitted the x because it is irrelevant in evaluating the limit. Since the limits from the left hand side and the right hand side are equal, the limit exists and is zero by the Squeeze theorem:

$$\lim_{(x,y)\to(0,0)} y^3 e^{-y^2} \approx \lim_{(x,y)\to(0,0)} \frac{y^3 e^{-y^2}\sin(xy)}{xy} = 0. \tag{8.15}$$

9. Consider the heat distribution along a single axis $U(x,t)$ (such as a wire). The distribution of heat over time is goverened by the heat equation,

$$\frac{\partial U}{\partial t} = \kappa \frac{\partial^2 U}{\partial^2 x},$$

where κ is the thermal diffusivity constant. You can solve for $U(x,t)$ given a set list of initial conditions (called "boundary conditions") on $U(x,t)$. Explain the meaning of the derivatives U_t and U_{xx} and how their spatial and temporal components are related.

For example, a relatively simple solution to the heat equation, for $0 \leq x \leq L$ and $t \geq 0$, is

$$U(x,t) = 6 \sin\left(\frac{\pi x}{L}\right) e^{-\kappa(\pi/L)^2 t}$$

where the explicit boundary conditions are $U(x,0) = 6\sin\left(\frac{\pi x}{L}\right)$, $U(0,t) = U(L,t) = 0$.

Solution: Let us consider the left hand and right hand sides of the equation seperately first. On the left hand side, is a first-order derivative in time, $\frac{\partial U(x,t)}{\partial t}$. This is a quantity defined at every point in time and along the spatial distribution axis. At every point in space (along x), there is a value of $\frac{\partial U(x,t)}{\partial t}$.

Since partial derivatives represent a rate of change of a scalar quantity along a specific axis, the time derivative therefore expresses how the heat magnitude at the point x is changing through time. This may be different at each point in space. Any partial differential equation which is first-order in time represents a *diffusion* equation. This implies that heat distributed along the spatial axis will tend to diffuse: hotter areas tend to grow colder, and colder areas tend to grow warmer, until equilibrium is reached. This is represented in the samply solution by the $e^{-\kappa(\pi/L)^2 t}$ term. As t increases from 0, the magnitude of heat along the whole axis will approach 0 (which would be equilibrium), since $\lim_{t\to\infty} e^{-at} = 0$.

Now consider the spatial derivative: $\frac{\partial^2 U(x,t)}{\partial x^2}$. You may recall that second derivatives contain information on the curvature of the function, that is, the rate of change of the rate of change along x. This also represents the rate of change of the slope for every point in time and space (x,t). The heat distribution itself is continuous (and differentiable - we're not simulating the Weierstrass function), which implies that both first and second derivatives are also continuous, hence as one decreases/increases in slope, so do the surrounding points (mutual influence).

Putting these ideas together, we can explain the heat equation. On the left hand side, we have a diffusion, which brings every point in space to zero as $t \to \infty$. On the right hand side, is how much the slope is changing at each point relative to the nearby points and time. These two are proportional to each other by the constant κ. Hence, all slopes have a tendency to approach a time when they stop changing: an equilibrium. Over time, they desire to find this equilibrium, so we will measure a decrease in overall curvature. A great website to simulate this is `https://visualize-it.github.io/heat_equation/simulation.html` .