MAT235 Tutorial 6

TA: Jace Alloway (j.alloway@mail.utoronto.ca)
Office Hours: M14-16 PG101

This week: Chain rule. Refer 14.5-14.6 Hughes-Hallett.

Chain Rule (Single-Variable): Involves taking the derivatives of nested functions. For $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ continuous differentiable to first order, then

$$\frac{d}{dx} f(g(x)) = \frac{df}{dg}\frac{dg}{dx}, \tag{1}$$

that is, the derivative of f with respect to g times the derivative of g with respect to x.

Chain Rule (Multi-Variable): Now consider continous, differentiable functions $f : \mathbb{R}^2 \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$, then the derivative of f with respect to x must be treated as a *total derivative*. This means that we cannot just simply take

$$\frac{d}{dx} f(g(x), h(x)) = \frac{df}{dg}\frac{dg}{dx}, \tag{2}$$

because it is not invariant under interchange $g \leftrightarrow h$. We must instead take the sum of all components (all 'paths' to reach the derivative), differentiating only with respect to each function:

$$\frac{d}{dx} f(g(x), h(x)) = \frac{\partial f}{\partial g}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial h}\frac{\partial h}{\partial x} \tag{3}$$

which enforces invariance under reordering of the inputs of f.

Equation of a Tangent Plane: Consider an implicit surface $f(x, y, z) = c$. If we have a curve $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$, or $\mathbf{r}(t)$, traced out on the surface of our function $f(\mathbf{r}(t))$, then the tangent vector of that curve will always be tangent to the surface of f. That is,

$$\frac{d}{dt} f(\mathbf{r}(t)) = 0. \tag{4}$$

By the chain rule, we find that

$$\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f(x, y, z) \cdot \frac{d\mathbf{r}}{dt} = 0. \tag{5}$$

Equation (10) implies that, by the dot product, for any vector lying on the surface $f(x, y, z)$ at a fixed point $\mathbf{p}$, the gradient vector and the vector on the surface are orthogonal. Hence $\nabla f(\mathbf{p})$ is a normal vector at a point $\mathbf{p}$ on the surface. Hence

$$\mathbf{n} = \langle f_x, f_y, f_z \rangle \tag{6}$$

forms a tangent plane to the surface. At a point $\mathbf{p} = (a, b, c)$, we can then write

$$f_x(\mathbf{p})(x - a) + f_y(\mathbf{p})(y - b) + f_z(\mathbf{p})(z - c) = 0. \tag{7}$$

Consider an explicit example. Let $g(x, y)$ be a function and let $f : \mathbb{R}^3 \to \mathbb{R}$ be an function where $f(x, y, z) = g(x, y) - z$ so that $f(x, y, z) = 0$ is the surface $g(x, y)$. The derivatives of g in the x and y directions are g_x and g_y, whose vectors at any (x, y) point are

$$\mathbf{a} \equiv \langle 1, 0, g_x(x, y) \rangle \tag{8}$$

$$\mathbf{b} \equiv \langle 0, 1, g_y(x, y) \rangle \tag{9}$$

(the derivatives are rise/run, so multiplying through gives the vector in each direction). We can find the normal vector spanned by these two vectors at (x, y) via the cross-product:

$$\mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{pmatrix} \tag{10}$$

$$= -g_x \mathbf{i} - g_y \mathbf{j} + \mathbf{k} \tag{11}$$

hence, from our standard form equation for a plane, the tangent plane to a point (a, b, c) with $c = g(a, b)$ is

$$g_x(a, b)(x - a) + g_y(a, b)(y - b) - 1(z - g(a, b)) = 0. \tag{12}$$

Equivalently, if we were to use (7), with $g(x, y) - z = 0$,

$$\nabla f(x, y, z) = \langle g_x, g_y, -1 \rangle \tag{13}$$

thus the equation of our plane is identical to (12).

Problems:

(1) Suppose $z = f(x, y)$ and let $x = s + t$ and $y = s - t$. Simplify the expression

$$\left(\frac{\partial z}{\partial s}\right)\left(\frac{\partial z}{\partial t}\right)$$

in terms of f_x and f_y.

(2) Suppose $f(x, y)$ is a function of x and y, and define $g(u, v) = f(e^u + \cos v,\ e^u + \sin v)$. Find $g_u(0, 0)$ and $g_v(0, 0)$ given

Point	$f(x,y)$	$f_x(x,y)$	$f_y(x,y)$	$g(x,y)$
$(0,0)$	2	2	3	4
$(2,1)$	4	1	-3	-1

(3) Consider the surface $x^2 + y^2 + z = 0$. Find all points (a, b, c) on the surface such that the tangent plane at (a, b, c) has the normal vector $\langle 1, -1, 1\rangle$.