MAT235 Tutorial 7

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This week: Term Test 2 (Friday, November 29) review. Refer 14.2-14.8 and 15.1 Hughes-Hallett. Some notes on quadratic Taylor approximations and mixed second-order partials.

Quadratic Taylor Polynomials: For functions $f : \mathbb{R}^n \to \mathbb{R}$, the quadratic Taylor polynomial can be expanded in terms of the gradient $\nabla f(\mathbf{a})$ and the Hessian matrix

$$Hf(\mathbf{a}) = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{pmatrix}\Bigg|_{\mathbf{a}} \tag{1}$$

which is the $n \times n$ matrix of second-order partial derivatives of f. The expansion about a point $\mathbf{a}$ is

$$P_2(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a})(\mathbf{x}-\mathbf{a}) + \frac{1}{2}(\mathbf{x}-\mathbf{a})^T Hf(\mathbf{a})(\mathbf{x}-\mathbf{a}) \tag{2}$$

where $(\mathbf{x}-\mathbf{a})^T$ is the transpose of $(\mathbf{x}-\mathbf{a})$, and the vector products imply matrix multiplication. For a function $g : \mathbb{R}^2 \to \mathbb{R}$ around the point (a, b), one obtains

$$\begin{aligned} P_2(x,y) &= g(a,b) + \nabla g(a,b)[(x,y)-(a,b)] + \frac{1}{2}[(x,y)-(a,b)]^T Hg(a,b)[(x,y)-(a,b)] \qquad (3) \\ &= g(a,b) + g_x(a,b)(x-a) + g_y(a,b)(y-b) \\ &\qquad + \frac{1}{2}g_{xx}(x-a)^2 + g_{xy}(a,b)(x-a)(y-b) + \frac{1}{2}g_{yy}(a,b)(y-b)^2 \qquad (4) \end{aligned}$$

Clairaut's Theorem: Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function which is continuous differentiable up until, at least, second-order (this is called a C^2 function: all second-order derivatives of f are continuous, assuming f is differentiable up until second-order). Then, for any sequence of applied partial derivatives on f, one has

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i} \tag{5}$$

where x^i and x^j represent coordinate variables, for $0 \le i, j \le n$.
What this means is that, as long as the second-order derivatives of f are continuous, the partial derivatives along axes commute and can be applied in any order. For example, let $h : \mathbb{R}^2 \to \mathbb{R}$ be given as $h(x, y) = \sin(xy)$. Here, h is a continuous function, and the second-order derivatives of h are

$$\begin{aligned} h_{xx} &= -y^2\sin(xy) \qquad & h_{yy} &= -x^2\sin(xy) \\ h_{xy} &= -xy\sin(xy) \qquad & h_{yx} &= -xy\sin(xy) \end{aligned} \tag{6}$$

which are also continuous on all of $\mathbb{R}^2$. By Clairaut's theorem, one may observe that $h_{xy} = h_{yx}$, even though the order of partial derivatives is exchanged.

Thus, *if the function is continuous differentiable up until second-order, the order of partial derivatives does not matter - you will still arrive at the same answer.* It is for this reason there is not a

$\frac{1}{2}$ factor in front of the g_{xy} term in (4), because $g_{xy} = g_{yx}$ and we sum. Using Clairaut, one can conclude that the Hessian is a symmetric matrix.

Problems (taken up in tutorial):

1. Suppose $f(2,1) = 5$, $f_x(x,y) = 6xy + y^2 e^{xy^2}$ and $f_y(x,y) = 3x^2 + 2xye^{xy^2}$. Find the quadratic Taylor polynomial of f at (2.1). Use it to approximate $f(2.1, 1.1)$.

2. Suppose that

$$\begin{aligned} f_x(2,1) &= 2.2 & \qquad f_y(2.1) &= -0.8 \\ f_x(2.5,1) &= 1 & \qquad f_y(2.5,1) &= -1.2 \\ f_x(2,1.5) &= 1.8 & \qquad f_y(2,1.5) &= -1.4. \end{aligned}$$

   If $f(2,1) = 0$, estimate the value of $f(2.05, 0.9)$ using a quadratic Taylor polynomial about (2,1). Use difference quotients to approximate all second-order partial derivatives.

3. The heat equations is the partial differential equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, where $u = u(x,t)$.
   (a) Show that $u(x,t) = e^{-\pi^2 t}\sin(\pi x) + \pi$ is a solution to the heat equation.
   (b) If $u(x,t) = e^{at}\sin(bx)$ satisfies the heat equation, find the relationship between a and b.

4. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a function defined by $f(x_1, x_2, \ldots, x_n) = \exp(a_1x_1 + a_2x_2 + \ldots a_nx_n)$ under the constraint $a_1^2 + a_2^2 + \ldots a_n^2 = 1$. Simplify the expression

$$f_{x_1x_1} + f_{x_2x_2} + \cdots + f_{x_nx_n}.$$

Additional Problems: (solutions will be posted by Wednesday next week):

5. Let $g : \mathbb{R}^2 \times [0,\infty)^2 \to \mathbb{R}$ be a function defined by

$$g(x,y,z,w) = \int_w^z \frac{\arctan(x^2)}{1+u^u}du$$

   where $x, y \in \mathbb{R}$ but $z, w \geq 0$. Compute the following partial derivatives of g:

$$\frac{\partial g}{\partial x}, \quad \frac{\partial g}{\partial y}$$

$$\frac{\partial^2 g}{\partial w^2}, \quad \frac{\partial^2 g}{\partial z^2}, \quad \frac{\partial^2 g}{\partial w \partial z}, \quad \frac{\partial^2 g}{\partial z \partial w}$$

6. Determine the Quadratic Taylor polynomial $P_2(x,y)$ of the function $j : \mathbb{R}^2 \to \mathbb{R}$ given by $j(x,y) = \arctan(x+2y)$ centered about the point (1,0).

7. Let $q(x,y,z) = \ln(x^2 + \cos(y) + e^{z^2} + 2)$. Find the second order Taylor polynomial of q about $(1, 3\pi/4, 0)$.

8. Consider a particle in an infinitely deep potential well of width L along $[0,L]$. The time-dependent wavefunction is given by $\psi(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{iE_n t}{\hbar}}$ where the c_n are coefficients, $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$ is the energy of the particle at the n-th excitation, $\sqrt{\frac{2}{L}}$ yields probabilistic normalization, and m is the mass of the particle. Show that $\psi(x,t)$ satisfies the Schrödinger equation on the domain $0 \leq x \leq L$,

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t).$$