MAT235 Tutorial 8

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This week: Double integrals and iterated integrals over bounded domains. Refer 16.1-16.3 Hughes-Hallett.

"Even and Odd" Function Properties: Consider a function $f : \mathbb{R} \to \mathbb{R}$ in one dimension. f is called an *even* function if it, when reflected along the y-axis, remains invariant. That is,

$$f(-x) = f(x) \iff f \text{ even.} \tag{1}$$

Similarly, f is called an *odd* function if it, when reflected along the y axis and the x axis, remains invariant. That is,

$$f(-x) = -f(x) \iff f \text{ odd.} \tag{2}$$

These properties are useful for evaluating integrals, as a definite integral is defined as the area between the x axis and the curve. Let $g : \mathbb{R} \to \mathbb{R}$ be a function defined on a symmetric interval $A = [-a, a]$ around 0 where $a > 0$ is a real number (or unbounded). Consider the integral

$$I = \int_A g(x)\,dx. \tag{3}$$

We obtain the following properties from (1) and (2):

$$g \text{ even } \implies I = 2\int_0^a g(x)\,dx; \tag{4}$$

$$g \text{ odd } \implies I = 0. \tag{5}$$

You can think about this intuitively, as how the positive-negative area changes from one side of the interval to the other and how is behaves under continuous addition.

Integrals over Rectangles: One of the most common problems encountered in multivariable integration are the rules involving exchanging an integration order in an $n > 1$-dimensional integral. This is particularly important for when one computes an integral over a rectangular surface, or some other arbitrary shape (such as a functional-defined boundary). As we will work with an example using a triangle (there is no generalization for evaluating iterated integrals), I will briefly note an important theorem for which integrals are taken out over a rectangle.
<u>Theorem (Fubini)</u>: Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a function which is integrable on a rectangle $A \times B \subseteq \mathbb{R}^2$ (A and B are intervals in $\mathbb{R}$). Then, one may evaluate the double integral $\iint_{A\times B} h(x, y)\,dA$ as an iterated integral:

$$\iint_{A\times B} h(x,y)\,dA = \int_A \left[\int_B h(x,y)\,dy\right] dx = \int_B \left[\int_A h(x,y)\,dx\right] dy, \tag{6}$$

that is, one may compute an integral along either dimension first, then integrate along the other. In MAT235, we can say that a function is integrable if it is continuous along respective axes of integration, though in reality one must consider the Lebesque integral.

Review of Single-Variable Integration Methods:

- *u-substitution:* Let $u = f(x)$ be a variable/functional substitution so that the integral becomes easier to compute. Then $du = \dfrac{df}{dx}\,dx$, and the bounds of the integral change according to $u_a = f(a)$ and $u_b = f(b)$.

- *integration by parts:* reverse product rule. One has

$$\int_a^b u dv = uv\big|_a^b - \int_a^b v du, \tag{7}$$

where $u = u(x)$ and $v = v(x)$ and du, dv are the differentials.

- *indefinite integrals:* For the integral $\displaystyle\int_0^\infty f(x)\,dx$, one must compute $\displaystyle\lim_{a\to\infty}\int_0^a f(x)dx$.

- *Feynman integration:* differentiation under the integral sign. Let $I = \displaystyle\int_a^b f(x)dx$. Introduce a parameter t to the integral, so that $I = I(t = C)$, and we integrate $I(t) = \displaystyle\int_a^b f(x,t)dx$. One may choose t arbitrarily so that we integrate

$$I'(t) = \frac{d}{dt}\int_a^b f(x,t)dx = \int_a^b \frac{\partial f(x,t)}{\partial t}dx. \tag{8}$$

If $I'(t)$ can be integrated, then we apply a further integral in t so that $I(t = C) = \displaystyle\int_0^C I'(u)du$. The trick is to choose t in an appropriate context so that we can integrate $I'(t)$ in x, then in t. This is not required for MAT235.

For more information, see https://zackyzz.github.io/feynman.html.

Problems:

1. Let $R = [-1, 1] \times [1, \pi/2]$. Evaluate the double integral
$$\iint_R \frac{\sin x \cos y}{y(2+3x^4)} dA. \tag{9}$$
Hint: see notes on even and odd functions. Split the integrand into a product $g(x)h(y)$.

2. Consider the iterated integral
$$\int_0^2 \int_{2x}^4 \frac{y}{y^3+1}\, dy\, dx. \tag{10}$$
Sketch the region of integration and evaluate the integral.

3. Find an expression in the form of an iterated integral which gives the volume of the solid E bounded by the cylinders $z = y$ and $x = 1 - y^2$ in the first octant $(x, y, z \geq 0.)$