MAT235 Tutorial 9

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This week: Change of variables. Refer 21.2 Hughes-Hallett.

Determinants: A determinant represents a scaling quantity in 2 or more dimensions. For a change of variables in $\mathbb{R}^2$, area is scaled via the determinant of the Jacobian. In $\mathbb{R}^3$, volume is scaled. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function. The Jacobian of f is a matrix full of all the first-order partial derivatives of f, where f is differentiable.

$$\mathbf{J}_f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \tag{1}$$

Determinants can only be taken if the matrix is $n \times n$: as many rows as columns. This is because dimension must be conserved over a change of area/volume/hyper-volume. For 2×2 matrices,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \tag{2}$$

while for 3×3 matrices,

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - gec - hfa - idb. \tag{3}$$

Hence, for every change of variables $\mathbf{f}(x, y) = (u, v)$, the area $dA = dxdy$ scales as $dA' = |\det[\mathbf{J}_f]|dudv$ (the absolute values are for the absolute area scale). The scaling of the inverse transformation is $\dfrac{1}{|\det[\mathbf{J}_f]|}$.

Problems:

1. Evaluate
$$\iint_D \sin(9x^2 + 4y^2)\, dA,$$
where $D = \{(x, y) \in \mathbb{R}^2 \,:\, 9x^2 + 4y^2 \leq 1\}$ (the region bounded by the ellipse). *Hint: Perform a change of variables to integrate over a circle.*

2. Let R be the region in the first quadrant bounded by the lines $y = x$, $y = 3x$ and the curves $xy = 1$, $xy = 3$. Using the change of variables $x = u$, $y = \frac{v}{u}$, re-write the integral
$$\iint_R xy\, dA$$
in the (u, v) coordinate system with integration order $dudv$.